Map of Common Knowledge Logics*

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January 30, 2001

Abstract

In order to capture the concept of common knowledge, various extensions of multi-modal epistemic logics, such as fixed-point type extensions and infinitary ones, have been proposed. Although we have now a good list of such proposed extensions, the relationships among these extensions are still unclear. The purpose of this paper is to draw a map showing the relationships among them. In the propositional case, these logics turn to be all Kripke complete and could be comparable in a meaningful manner. In the predicate case, there is the gap shown by F. Wolter that the predicate extension of the Halpern-Moses fixed-point type common knowledge logic is Kripke incomplete. However, if we go further to an infinitary extension, Kripke completeness would be recovered. In drawing the map, we focus on what are happening around the gap in the predicate case. The map enables us to better understand the common knowledge logics as a whole.

1. Introduction

Multi-agent epistemic logics have been developed for investigations of multi-agents interactions such as game theoretical problems. In such multi-agent situations, common knowledge is important in discussing knowledge (or beliefs) shared among agents. Various extensions have been proposed in order to capture the concept of common knowledge. These extensions are divided into two types: the *fixed-point* and *infinitary* approaches, e.g., Halpern-Moses [4] is in the former and Kaneko-Nagashima [8], [9] in the latter. Some of them are given as propositional logics and some others as predicate logics. Also, some are considered from the model-theoretic viewpoint and some others from the proof-theoretic viewpoint. Thus, we have a good list of extensions of epistemic logics, but their mutual relationships are yet unclear.

*The authors are partially supported by Grant-in-Aids for Scientific Research No.106330003, No.09740131 and No.10440027, Ministry of Education, Science and Culture. The third author is supported also by the Sumitomo Foundation.

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The purpose of this paper is to draw a map showing the relationships among those extensions.

In the propositional case, these extensions turn to be all Kripke complete, and are comparable meaningfully in the sense that if one is an extension of another, it is also a *conservative* extension. In contrast to the propositional case, Wolter [21] proved in the predicate case that the set of valid formulae in the Kripke semantics is not recursively enumerable in the presence of common knowledge. This has the implication that any finitary predicate extension of an epistemic logic cannot capture the Kripke semantics with common knowledge. In other words, the latter has no finitary proof theory. Nevertheless, it is also known from Tanaka-Ono [19] and Tanaka [16] that this difficulty does not occur in the infinitary approach. Thus, in the predicate case, there is some gap from the fixed-point logic approach to the infinitary approach. In drawing a map of common knowledge logics, we will focus especially on what are happening around this gap.

Diagram 1.1 gives some extant extensions of multi-agent epistemic logics. Since we discuss various common knowledge extensions, we adopt KD4-type axioms as basic epistemic axioms on belief operators. Thus, we will start with propositional multi-agent epistemic logic $\mathrm{KD4}^n$, where n is the number of agents. The prefix Q means the predicate extension of a propositional logic together with the Barcan axiom. For example, $\mathrm{QKD4}^n$ is the predicate extension of $\mathrm{KD4}^n$. Logic HM is the fixed-point type extension of $\mathrm{KD4}^n$ due to Halpern-Moses [4], where the common knowledge of a formula is determined by adding one axiom and one inference rule to $\mathrm{KD4}^n$. Logic GL_ω is an infinitary extension of $\mathrm{KD4}^n$ due to Kaneko-Nagashima [8] and [9], where the common knowledge of a formula is explicitly expressed as an infinitary conjunctive formula, and QGL_ω is its predicate extension.[1]

	Finitary Base Logic	Finitary	Infinitary
Propositional	$\mathrm{KD4}^n$	HM	GL_ω
Predicate	$\mathrm{QKD4}^n$	QHM	QGL_ω

Diagram 1.1

The Kripke completenesses of $\mathrm{KD4}^n$ and $\mathrm{QKD4}^n$ have been known as variants of completeness results given in modal logic literature (cf., Hughes-Cresswell [6]). It is also known from Halpern-Moses [4] (see also Lismont-Mongin [11], Fagin, *et.al* [1] and Meyer-van der Hoek [12]) that HM is complete. It follows from Tanaka-Ono [19] that GL_ω is Kripke complete. These completeness results in the propositional case imply that if one logic is an extension of another, then it is a conservative extension. On the other hand, it is known in the predicate case that there is a gap from QHM to QGL_ω. As already mentioned, $\mathrm{QKD4}^n$ is Kripke complete, and so is QGL_ω, which could be expected by Tanaka-Ono [19] and Tanaka [18]. However, it

[1] We may find some other approach such as Segerberg [13]. Tanaka [17] discussed completeness of such a logic in the predicate case from the viewpoints of noncompact logics.

follows from Wolter's [21] theorem that QHM is Kripke incomplete. Thus, the gap above mentioned exists between QHM and QGL_ω.

In the propositional case, the extensions are successfully made from HM to GL_ω. We have Kripke incompleteness only for QHM. This non-parallelism may be interpreted in two ways: One interpretation is that the predicate case is regarded as a conundrum. The other is that it is a conundrum to have such successful results in the propositional case, since a common knowledge extension includes implicitly infinite conjunctions. After all, we will not able to conclude which case would be a conundrum. Nevertheless, we will see approximately where it occurs.

From the syntactical point of view, the distance from QHM to QGL_ω is large in that QGL_ω allows infinitary conjunctions and disjunctions. To consider the question of where from QHM to QGL_ω the gap occurs, we provide other two logics, propositional CX and CY, and their predicate extensions QCX and QCY. In the propositional case, both CX and CY are shown to be equivalent to HM. In the predicate case, QCY is shown to be Kripke complete, which together other problems will be discussed in Tanaka [16]. This result together with Wolter's [21] theorem that implies that QCY is not equivalent to QHM. Diagram 1.2 gives a full list of logics to be considered in this paper. The gap occurs only from QHM to QCY. Logics HM and CY are faithfully embedded into infinitary GL_ω, which relation is denoted by $\rightsquigarrow$.

Propositional	$KD4^n$	$\rightarrow$	HM $\longleftrightarrow$	CX	$\longleftrightarrow$ CY		$\rightsquigarrow$	GL_ω
	$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$			$\downarrow$
Predicate	$QKD4^n$	$\rightarrow$	QHM $\rightarrow$	QCX	$\rightarrow$ QCY		$\rightsquigarrow$	QGL_ω
formulae	finitary		finitary	finitary				infinitary
proofs	finitary		finitary	infinitary				infinitary

Diagram 1.2

Logics CX, CY and QCX, QCY keep the set of finitary formulae in the same way as HM and QHM. However, the formers include infinitary inference rules, which implies that infinitary proofs are required. In this sense, QCX and QCY are located also between QHM and QGL_ω.

Logics CX and QCX look more natural than CY and QCY. The formers are defined by adding, to $KD4^n$ and $QKD4^n$, two axiom schemata and one inference rule directly on common knowledge. In contrast, logics CY and QCY are defined by strengthening the inference rule to a somewhat *artificial* rule. As already stated, Kripke completeness is available for CX, CY, QCY, and the Kripke completeness of QCX remains open.

We should mention another trial to avoid the gap. In contrast to our approach of considering extensions of QHM, Sturm-Wolter-Zakharyaschev [15] considered a fragment of QHM, which is the monodic fragment and defined in Section 5. They proved the Kripke completeness of the monodic fragment of QHM with no function symbols and no equality. This fragment is located between HM and QHM. The jump occurs after the monodic fragment of QHM.

Tanaka [16] considers the behavior of logic QCY in details as well as its Kripke completeness in the case of no function symbols. The original motivation for predicate common knowledge logics is to consider theories with interactions between agents such as game theoretical problems (cf., Kaneko-Nagashima [8]). Hence, it is preferable to include function symbols as well as equality in such a first-order theory. A proof of the completeness theorem for QCY can be obtained by modifying Tanaka's [16] proof of the completeness of QCY.

We will give a unified treatment of these logics. We need to mention paralleled results for propositional and predicate cases. However, we would often mention results only in the predicate case since they can be stated in the propositional case without much difficulty.

2. Logics KD4^n and QKD4^n

In this section, we formulate propositional KD4^n and predicate QKD4^n so that they are directly comparable. Common knowledge logics will be defined as extensions of these logics. In contrast with the multiplicity of syntactical systems, the Kripke semantics is uniquely defined and enables us to make comparisons of various syntactical systems. As stated in Section 1, we will make a choice of KD4-type logical axioms throughout the paper.

2.1. Language

Throughout the paper, we use the following list of symbols:

Free variables: $\mathbf{a}_0, \mathbf{a}_1, ...$; *Bound variables*: $\mathbf{x}_0, \mathbf{x}_1, ...$;

Logical connectives: $\neg$ (*not*), $\supset$ (*implies*), $\wedge$ (*and*), $\vee$ (*or*);

Quantifiers: $\forall$ (*for all*), $\exists$ (*exists*);

Function symbols: $\mathbf{f}_0, \mathbf{f}_1, ...$; *Predicate symbols*: $\mathbf{P}_0, \mathbf{P}_1, ...$;

Unary belief operator symbols: $\mathrm{B}_1, ..., \mathrm{B}_n$;

Unary common knowledge operator symbol: C;

Parentheses: (,).

The subscripts $1, ..., n$ of $\mathrm{B}_1, ..., \mathrm{B}_n$ are the names of *agents*. We denote the set of agents $\{1, 2, ..., n\}$ by N. In the following, we consider the case of $n \geq 2$. We denote the set of all free variables by FV. We assume that there are countably infinite numbers of free variables and bound variables. Each $\mathbf{f}_k$ is assumed to be an *l-ary* function symbol for some $l \geq 0$, and each $\mathbf{P}_k$ is an l-ary predicate symbol for some $l \geq 0$. When $l = 0$, $\mathbf{f}_k$ is a *constant* symbol and $\mathbf{P}_k$ is a *propositional* variable. We denote the list of these function and predicate symbols by $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...]$. We assume that there is at least one 0-ary predicate symbol but there may be no function symbols.

Terms and *formulae* are defined in the standard finitary manners. We denote the set of all formulae by $\mathcal{P}$. We denote, by $\mathcal{P}_{-\mathrm{C}}$, the set of formulae in $\mathcal{P}$ which have no occurrences of C, and by $\mathcal{P}_{-\mathrm{B}}$, the set of formulae in $\mathcal{P}$ which have no occurrences of $\mathrm{B}_1, ..., \mathrm{B}_n$. The set of formulae including neither C nor $\mathrm{B}_1, ..., \mathrm{B}_n$ is

$\mathcal{P}_{-\mathrm{BC}} = \mathcal{P}_{-\mathrm{B}} \cap \mathcal{P}_{-\mathrm{C}}$. We say that a formula A (or a term t) is *closed* iff no free variable occurs in A (respectively, in t).

We define the *propositional fragment* $\mathcal{P}_{-\mathrm{Q}}$ of $\mathcal{P}$ to be the set of formulae generated from the 0-ary predicate symbols without quantifiers. We also denote $\mathcal{P}_{-\mathrm{C}} \cap \mathcal{P}_{-\mathrm{Q}}$ and $\mathcal{P}_{-\mathrm{BC}} \cap \mathcal{P}_{-\mathrm{Q}}$ by $\mathcal{P}_{-\mathrm{CQ}}$ and $\mathcal{P}_{-\mathrm{BCQ}}$. The set $\mathcal{P}_{-\mathrm{CQ}}$ is the propositional fragment without including common knowledge operator C, and is used to define $\mathrm{KD4}^n$.

For any $A \in \mathcal{P}$, we define the set $\mathrm{Sub}(A)$ of subformulae of A inductively as follows:

(0) $\mathrm{Sub}(A) = \{A\}$ for any atomic formula A;

(1) $\mathrm{Sub}(\neg A) = \mathrm{Sub}(A) \cup \{\neg A\}$;

(2) $\mathrm{Sub}(A * B) = \mathrm{Sub}(A) \cup \mathrm{Sub}(B) \cup \{A * B\}$, where $*$ is $\supset, \wedge$ or $\vee$;

(3) $\mathrm{Sub}(QxA(x)) = \bigcup_{t \text{ is a term}} \mathrm{Sub}(A(t)) \cup \{QxA(x)\}$, where Q is $\forall$ or $\exists$;

(4) $\mathrm{Sub}(\mathrm{B}_i(A)) = \mathrm{Sub}(A) \cup \{\mathrm{B}_i(A)\}$ for $i \in N$;

(5) $\mathrm{Sub}(\mathrm{C}(A)) = \mathrm{Sub}(A) \cup \{\mathrm{C}(A)\}$.

We call B a *subformula* of A iff $B \in \mathrm{Sub}(A)$.

The set $\mathcal{P}_{-\mathrm{B}}$ is *subformula-closed*, i.e., if $A \in \mathcal{P}_{-\mathrm{B}}$ and B is a subformula of A, then $B \in \mathcal{P}_{-\mathrm{B}}$. The sets $\mathcal{P}_{-\mathrm{C}}$, $\mathcal{P}_{-\mathrm{BC}}$ and $\mathcal{P}_{-\mathrm{Q}}$ are also subformula-closed.

2.2. Epistemic Predicate Logic $\mathrm{QKD4}^n$ and its Propositional Fragment $\mathrm{KD4}^n$

We give the following seven axiom schemata and five inference rules: For any formulae A, B, C, and term t,

L1: $A \supset (B \supset A)$;

L2: $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

L3: $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;

L4: $A \wedge B \supset A$ and $A \wedge B \supset B$;

L5: $A \supset A \vee B$ and $B \supset A \vee B$;

L6: $\forall x A(x) \supset A(t)$;

L7: $A(t) \supset \exists x A(x)$,

and

$$\frac{A \supset B \quad A}{B} \text{ (MP)}$$

$$\frac{A \supset B \quad A \supset C}{A \supset B \wedge C} \ (\wedge\text{-Rule}) \qquad \frac{A \supset C \quad B \supset C}{A \vee B \supset C} \ (\vee\text{-Rule})$$

$$\frac{A \supset B(a)}{A \supset \forall x B(x)} \ (\forall\text{-Rule}) \qquad \frac{A(a) \supset B}{\exists x A(x) \supset B} \ (\exists\text{-Rule}),$$

where the free variable a does not occur in the lower formulae of $\forall$-Rule and $\exists$-Rule.

The above logical axioms and inference rules form classical logic. We write the set of L1–L5 and MP, $\wedge$-Rule and $\vee$-Rule by PCL, and the set of all axioms and inference rules by QCL. In fact, the choice of a set of formulae is still needed to determine a logic. In the following, *classical propositional logic* is understood to be PCL within $\mathcal{P}_{-\mathrm{BCQ}}$, and *classical predicate logic* is QCL within $\mathcal{P}_{-\mathrm{BC}}$.

The following are axioms and inference rule for belief operators B_i for $i = 1, ..., n$:

K: $\mathrm{B}_i(A \supset C) \supset (\mathrm{B}_i(A) \supset \mathrm{B}_i(C))$;

D: $\neg\mathrm{B}_i(\neg A \wedge A)$;

4: $\mathrm{B}_i(A) \supset \mathrm{B}_i\mathrm{B}_i(A)$;

$\forall$-B: $\forall x \mathrm{B}_i(A(x)) \supset \mathrm{B}_i(\forall x A(x))$;

and

$$\text{Necessitation:} \quad \frac{A}{\mathrm{B}_i(A)}.$$

In literature, 4 and $\forall$-B are called, respectively, the *Positive Introspection* axiom and *Barcan* axiom. Throughout this paper, we assume the Barcan Axiom in the predicate case.

Propositional $\mathrm{KD4}^n$ and predicate $\mathrm{QKD4}^n$ are defined as follows:

$\mathrm{KD4}^n$: PCL + (K + D + 4 + Nec) within $\mathcal{P}_{-\mathrm{CQ}}$;

$\mathrm{QKD4}^n$: QCL + (K + D + 4 + $\forall$-B + Nec) within $\mathcal{P}_{-\mathrm{C}}$.

In $\mathrm{KD4}^n$, any formulae in the axioms and inference rules are restricted to $\mathcal{P}_{-\mathrm{CQ}}$. On the other hand, in $\mathrm{QKD4}^n$, those are restricted to $\mathcal{P}_{-\mathrm{C}}$. When we employ $\mathcal{P}_{-\mathrm{CQ}}$, Axioms L6, L7 and $\forall$-B are automatically excluded, but when we do $\mathcal{P}_{-\mathrm{C}}$, these reappear. We remark that all the instances of the axiom schemata for $\mathrm{KD4}^n$ are allowed as axioms for $\mathrm{QKD4}^n$.

A proof in these logics is defined in the standard manner. In $\mathrm{QKD4}^n$, for example, a *proof* of A in $\mathcal{P}_{-\mathrm{C}}$ is a finite tree satisfying the following properties: (1) a formula in $\mathcal{P}_{-\mathrm{C}}$ is associated with each node of the tree and A is associated with the root; (2) the formula associated with each leaf is an instance of the axiom schemata of $\mathrm{QKD4}^n$; and (3) adjoining nodes together with the associated formulae form an instance of the inference rules of $\mathrm{QKD4}^n$.

We say that A is *provable* in $\mathrm{KD4}^n$ iff there is a proof of A in $\mathrm{KD4}^n$, which is denoted by $\vdash_{\mathrm{KD4}^n} A$. Similarly, we define the provability relation $\vdash_{\mathrm{QKD4}^n}$ of $\mathrm{QKD4}^n$. Then it holds that for any A in $\mathcal{P}_{-\mathrm{CQ}}$,

$$\vdash_{\mathrm{KD4}^n} A \text{ implies } \vdash_{\mathrm{QKD4}^n} A. \tag{2.1}$$

In fact, the converse holds, too, which we will mention later as Corollary 2.3.

2.3. Kripke Semantics with Constant Domains

Contrary to the multiplicity of syntactical systems to be discussed in this paper, it suffices to consider only one semantics. This common semantics together with the (soundness-) completeness result for each system enables us to make direct comparisons of those syntactical systems. The common semantics is the Kripke semantics with constant domains. As we chose KD4-type logics, we are going to consider only serial and transitive accessibility relations.

Recall that the list of function and predicate symbols is given as $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...]$. Let M be a nonempty set. A *classical interpretation* $\mathcal{I} = [\tilde{f}_0, \tilde{f}_1, ...; \tilde{P}_0, \tilde{P}_1, ...]$ on M consists of *interpretations* $\tilde{f}_k$ and $\tilde{P}_k$, respectively, with the following conditions:

F1: the interpretation $\tilde{f}_k$ of each l-ary $\mathbf{f}_k$ is a function from M^l to M;

F2: the interpretation $\tilde{P}_k$ of each l-ary $\mathbf{P}_k$ is a function from M^l to $\{\top, \bot\}$ (when $l = 0$, $\tilde{P}_k$ is simply either $\top$ or $\bot$).

A *Kripke frame* $\mathcal{F} = (W; R_1, ..., R_n; M)$ is an $(n+2)$-tuple of a *set of possible worlds* W, *accessibility relations* $R_1, ..., R_n$ over W and a *domain* M of individual variables. We assume the following conditions:

K1: W is an arbitrary nonempty set;

K2: R_i is a subset of $W \times W$ for $i = 1, ..., n$;

K3: M is an arbitrary nonempty set.

An *interpretation* $\mathcal{I}$ is a function which assigns to each $w \in W$ a classical interpretation $\mathcal{I}(w) = [\tilde{f}_0, \tilde{f}_1, ...; \tilde{P}_0^w, \tilde{P}_1^w, ...]$ on M with the property that only the interpretations $\tilde{P}_k^w$ of predicate symbols $\mathbf{P}_k$ may depend upon w but the interpretations $\tilde{f}_k$ of function symbol $\mathbf{f}_k$ are constant over W. A *Kripke model* $\mathcal{M}$ is a pair $(\mathcal{F}, \mathcal{I})$ of a Kripke frame $\mathcal{F}$ and an interpretation $\mathcal{I}$. Since we restrict our attentions to KD4-type logics, *we assume throughout this paper* that each accessibility relation R_i is serial and transitive. That is, we consider only a Kripke frame $\mathcal{F} = (W; R_1, ..., R_n; M)$ where each R_i is serial and transitive.

As the interpretations $\tilde{f}_k$ of function symbols $\mathbf{f}_k$ are constant over W, we interpret free variables as independent of a possible world. Hence, we have the following simple definition: A function $\sigma : FV \to M$ is called an *assignment*. One assignment σ is used for all possible worlds in W.

Let a pair $(\mathcal{I}, \sigma)$ of an interpretation $\mathcal{I}$ and an assignment σ be given. The *valuation* $V(t : (\mathcal{I}, \sigma))$ of a term t is the function from the set of terms to M defined inductively by

T1: $V(\mathbf{a}_k, (\mathcal{I}, \sigma)) = \sigma(\mathbf{a}_k)$ for all $\mathbf{a}_k \in FV$;

T2: $V(\mathbf{f}_k(t_1, ..., t_l), (\mathcal{I}, \sigma)) = \tilde{f}_k(V(t_1, (\mathcal{I}, \sigma)), ..., V(t_l, (\mathcal{I}, \sigma)))$.

For any free variable a, we write $\sigma \underset{a}{=} \sigma'$ iff $\sigma(b) = \sigma'(b)$ for all $b \in FV \Leftrightarrow \{a\}$. Also, we denote the set of finite sequences $(i_1, ..., i_m)$ in N by N^*.[2] Note that the null sequence ϵ belongs to N^*. We say that $u \in W$ is *reachable from* w in a Kripke frame $\mathcal{F} = (W; R_1, ..., R_n; M)$ iff there is a finite sequence $\{w_1, ..., w_m\}$ $(m \geq 1)$ in W and $(i_1, ..., i_{m-1}) \in N^*$ such that $w = w_1, u = w_m$ and $(w_t, w_{t+1}) \in R_{i_t}$ for $t = 1, ..., m \Leftrightarrow 1$.

Let $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ be a Kripke model. Then we define the *valuation relation* $(\mathcal{M}, \sigma, w) \models$ inductively as follows:

E0: for any atomic formula $\mathbf{P}_k(t_1, ..., t_l)$,

[2] For a different purpose, it would be more convenient to adopt $N^{**} = \{(i_1, ..., i_m) \in N^* : i_t \neq i_{t+1}$ for $t = 1, ..., m-1\}$ than N^*.

$(\mathcal{M}, \sigma, w) \models \mathbf{P}_k(t_1, ..., t_l) \Longleftrightarrow \tilde{P}_k^w(V(t_1, (\mathcal{I}, \sigma)), ..., V(t_l, (\mathcal{I}, \sigma))) = \top;$

E1: $(\mathcal{M}, \sigma, w) \models \neg A \Longleftrightarrow (\mathcal{M}, \sigma, w) \not\models A;$

E2: $(\mathcal{M}, \sigma, w) \models A \supset B \Longleftrightarrow (\mathcal{M}, \sigma, w) \not\models A$ or $(\mathcal{M}, \sigma, w) \models B;$

E3: $(\mathcal{M}, \sigma, w) \models A \wedge B \Longleftrightarrow (\mathcal{M}, \sigma, w) \models A$ and $(\mathcal{M}, \sigma, w) \models B;$

E4: $(\mathcal{M}, \sigma, w) \models A \vee B \Longleftrightarrow (\mathcal{M}, \sigma, w) \models A$ or $(\mathcal{M}, \sigma, w) \models B;$

E5: $(\mathcal{M}, \sigma, w) \models \forall x A(x) \Longleftrightarrow (\mathcal{M}, \sigma', w) \models A(a)$ for all $\sigma' \underset{a}{=} \sigma;$

E6: $(\mathcal{M}, \sigma, w) \models \exists x A(x) \Longleftrightarrow (\mathcal{M}, \sigma', w) \models A(a)$ for some $\sigma' \underset{a}{=} \sigma;$

E7: $(\mathcal{M}, \sigma, w) \models \mathrm{B}_i(A) \Longleftrightarrow (\mathcal{M}, \sigma, v) \models A$ for any v with $(w, v) \in R_i;$

E8: $(\mathcal{M}, \sigma, w) \models \mathrm{C}(A) \Longleftrightarrow (\mathcal{M}, \sigma, u) \models A$ for all u reachable from w,

where a is the first variable not occurring in $\forall x A(x)$ (or $\exists x A(x)$) in E5 (or E6, respectively). We write $\mathcal{M} \models A$ iff $(\mathcal{M}, \sigma, w) \models A$ for all assignments σ and worlds $w \in W$.

The semantic valuation relation $(\mathcal{M}, \sigma, w) \models$ is completely determined by the valuations of subformulae.

It is known that E8 has the following equivalent formulation:

Lemma 2.1. In the definition of $(\mathcal{M}, \sigma, w) \models$, E8 can be replaced by

E8*: $(\mathcal{M}, \sigma, w) \models \mathrm{C}(A) \Longleftrightarrow (\mathcal{M}, \sigma, u) \models \mathrm{B}_e(A)$ for all $e \in N^*$,

where $\mathrm{B}_e(A)$ is an abbreviation of $\mathrm{B}_{i_1}\mathrm{B}_{i_2}...\mathrm{B}_{i_m}(A)$ for $e = (i_1, ..., i_m) \in N^*$.

Condition E8* describes the intuitive understanding of the common knowledge of A that A is true, each player believes A, each believes that each believes A, and so on. If one wants to formulate "common beliefs", it is defined as the formula $\mathrm{B}_1\mathrm{C}(A) \wedge ... \wedge \mathrm{B}_n\mathrm{C}(A)$. Therefore, we focus on common knowledge in this paper.

The following completeness result has been known in literature (cf., Hughes-Cresswell [6]).

Theorem 2.2.(1)(Completeness for KD4^n). Let A be a formula in $\mathcal{P}_{-\mathrm{CQ}}$. Then $\vdash_{\mathrm{KD4}^n} A$ if and only if $\mathcal{M} \models A$ for all Kripke models $\mathcal{M} = (\mathcal{F}, \mathcal{I})$.

(2)(Completeness for QKD4^n). Let A be a formula in $\mathcal{P}_{-\mathrm{C}}$. Then $\vdash_{\mathrm{QKD4}^n} A$ if and only if $\mathcal{M} \models A$ for all Kripke models $\mathcal{M}$.

The converse of (2.1) is a simple consequence from Theorem 2.2, which we write down explicitly, since the same type of comparisons will be made throughout the paper.

Corollary 2.3 (Conservativity of QKD4^n upon KD4^n). Let A be a formula in $\mathcal{P}_{-\mathrm{CQ}}$. Then $\vdash_{\mathrm{KD4}^n} A$ if and only if $\vdash_{\mathrm{QKD4}^n} A$.

Before going to the next section, we should state Wolter's [21] result, which is one of the main concerns of this paper. We write $\models A$ iff $\mathcal{M} \models A$ for all Kripke models $\mathcal{M}$.

Theorem 2.4 (Non-Recursive-Enumerability). Suppose that the language $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...]$ contains at least nine unary predicate symbols. Then the set $\{A \in \mathcal{P} : \models A\}$ is not recursively enumerable.

This theorem does not depend upon the particular choice of the assumptions of transitivity and seriality on frames. It would remain to hold even if we strengthen the assumptions for frames to, for example, the S5-assumption that each R_i is an equivalence relation. For details, see Wolter [21].

Throughout the remaining part of this paper to avoid repetitive qualifications, we *assume* that the language $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...]$ has at least nine unary predicate symbols. Wolter proved in [20] the above non-recursive-enumerability result under this assumption, though Wolter [21] gave a different proof under the assumption of an infinite countable number of unary predicate symbols.

The set of provable formulae in $\mathrm{QKD4}^n$ is recursively enumerable, since $\mathrm{QKD4}^n$ is recursively axiomatizable. Therefore, it follows from Theorem 2.2 that the set $\{A \in \mathcal{P}_{-\mathrm{C}} : \models A\}$ is recursively enumerable (and so is $\{A \in \mathcal{P}_{-\mathrm{CQ}} : \models A\}$). Therefore, Theorem 2.4 is a phenomenon caused by introducing common knowledge into $\mathrm{QKD4}^n$. In subsequent sections, we will discuss more exactly when such a phenomenon occurs.

3. Common Knowledge Logics HM and QHM

Halpern-Moses [4] extended various multi-agent propositional epistemic logics into fixed-point logics to incorporate common knowledge. A variant is the fixed-point extension of $\mathrm{KD4}^n$. In this paper, since we focus on the KD4-type logics, we give the name HM to the fixed-point extension of $\mathrm{KD4}^n$, and QHM to the predicate extension of HM. In fact, it follows from Wolter's non-recursive enumerability theorem that QHM is Kripke incomplete. We are going to consider how the incompleteness result should be understood.

Consider the following axiom schema and inference rule:

CA: $\mathrm{C}(A) \supset A \wedge \mathrm{B}_1\mathrm{C}(A) \wedge ... \wedge \mathrm{B}_n\mathrm{C}(A)$;

CI: $$\frac{D \supset A \wedge \mathrm{B}_1(D) \wedge ... \wedge \mathrm{B}_n(D)}{D \supset \mathrm{C}(A)}.$$

Logics HM and QHM are defined, respectively, as follows:

HM: $\mathrm{KD4}^n$ + (CA+CI) within $\mathcal{P}_{-\mathrm{Q}}$;

QHM: $\mathrm{QKD4}^n$ + (CA+CI) within $\mathcal{P}$.

Axiom CA is often called the *fixed-point property* (with respect to A). It follows from this axiom in QHM (as well as in HM) that $\mathrm{C}(A)$ implies $\mathrm{B}_e(A) = \mathrm{B}_{i_1}...\mathrm{B}_{i_m}(A)$ for all $e = (i_1, ..., i_m) \in N^*$, which is explicitly written as Lemma 3.1. That is, these derived formulae show rather the intended meaning of "common knowledge". Inference rule CI means that if a formula D has the fixed-point property with respect to A, then D implies $\mathrm{C}(A)$, that is, $\mathrm{C}(A)$ is the deductively weakest formula having the fixed-point property. Inference rule CI is called the *fixed-point rule*. Logics HM and QHM are defined by the addition of the same axiom schema and inference rules to $\mathrm{KD4}^n$ within $\mathcal{P}_{-\mathrm{Q}}$ and to $\mathrm{QKD4}^n$ within $\mathcal{P}$.

Our concern is to consider the predicate extension QHM, but not HM. Therefore, we will mention properties only on QHM, and if some is need to make comparisons with HM, we would mention it also on HM.

Lemma 3.1. $\vdash_{\mathrm{QHM}} \mathrm{C}(A) \supset \mathrm{B}_e(A)$ for all $e \in N^*$.

The completeness theorem was proved for HM by Halpern-Moses [4].

Theorem 3.2 (Completeness for HM). For any $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{HM}} A$ if and only if $\models A$.

Let us return to predicate QHM. It is straightforward to see that QHM is sound with respect to the Kripke semantics. However, it is incomplete, which is an implication of Wolter's non-recursive enumerability theorem (Theorem 2.4). Indeed, since CA and CI as well as the others of QKD4^n are finitary, the set $\{A \in \mathcal{P} : \vdash_{\mathrm{QHM}} A\}$ is recursively enumerable. However, the set $\{A \in \mathcal{P} : \models A\}$ is not by Theorem 2.4. We summarize these results as the following theorem.

Theorem 3.3 (1)(Soundness). For any $A \in \mathcal{P}$, if $\vdash_{\mathrm{QHM}} A$, then $\models A$.

(2)(Incompleteness). There exists a formula $A \in \mathcal{P}$ such that $\models A$ but $\nvdash_{\mathrm{QHM}} A$.

The above incompleteness result does not rely upon a particular choice of a KD4-type logic. Indeed, Theorem 2.4 may be stated in a more elaborated manner. It states that incompleteness would remain even if we strengthen the logic by adding standard propositional axioms. For this, see Wolter [21].

We may see the difference between the above two theorems from two viewpoints: finitary viewpoint and infinitary viewpoint.

First, let us see the above theorems from the finitary point of view. As stated in Section 2, both KD4^n and QKD4^n are Kripke complete and have recursively enumerable sets of provable formulae. In the propositional case, HM is complete and $\{A \in \mathcal{P}_{-\mathrm{Q}} : \models A\}$ is also recursively enumerable, while in the predicate case, QHM is incomplete and $\{A \in \mathcal{P} : \models A\}$ is not recursively enumerable. Although HM and QHM are obtained from KD4^n and QKD4^n, respectively, by adding both finitary axiom schema CA and inference rule CI, only logic QHM turns to be incomplete. Hence this incompleteness is an unexpected jump. From the finitary point of view, the incompleteness of QHM may be regarded as a conundrum of making an unexpected jump.

Second, let us see the above theorems from the viewpoint of infinitary logics. In the infinitary approach, various completeness results are known, which will be discussed in Sections 4 and 6. Indeed, Kripke completeness is recovered in the sense that the strengthened provability in an infinitary approach captures $\{A \in \mathcal{P} : \models A\}$. In the propositional case, Kaneko [7] proved that HM can be regarded as a fragment of infinitary propositional epistemic logics. Hence we can regard HM as well as QHM as having already an infinitary aspect in part. The non-recursive-enumerability of $\{A \in \mathcal{P} : \models A\}$ is better understood from this point view. Hence, from the infinitary point of view, the completeness of HM may be regarded rather as a conundrum.

Before going to the next section, we mention two other lemmas on QHM and HM in order to relate them to other logical systems. The first is needed to motivate the two other common knowledge logics given in Section 4 . The second will be used to consider the C-fragment of QHM, which is closely related to the predicate extension of uni-modal S4.

The first lemma is stated as a derived inference rule in the semantic validity $\models$. We omit the proof of the lemma.

Lemma 3.4. If $\models D \supset \mathrm{B}_e(A)$ for all $e \in N^*$, then $\models D \supset \mathrm{C}(A)$.

By the completeness for HM (Theorem 3.2), we restate this lemma as follows: for $A, D \in \mathcal{P}_{-\mathrm{Q}}$,

($\mathbf{0}_{\mathrm{HM}}$): if $\vdash_{\mathrm{HM}} D \supset \mathrm{B}_e(A)$ for all $e \in N^*$, then $\vdash_{\mathrm{HM}} D \supset \mathrm{C}(A)$.

Lemma 3.1 and this claim describe the intended meanings of common knowledge. Therefore, it looks natural to define a common knowledge logic by adding these to $\mathrm{KD4}^n$ and $\mathrm{QKD4}^n$. However, since the infinite conjunction is implicit in common knowledge operator C, the Barcan property of belief operator B_i with respect to C is a problem. This Barcan property is captured by assuming $\mathrm{C}(A) \supset \mathrm{B}_i\mathrm{C}(A)$ for $i = 1, ..., n$. In the propositional case, we would obtain a logic equivalent to HM by adding, to $\mathrm{KD4}^n$, the instances of Lemma 3.1, inference rule 0_{HM} and the Barcan axiom for each B_i with respect to C. In the predicate case, we need stronger inference rules than 0_{HM}. These are the subjects of the next section.

In order to consider the question of what kind of formulae make a discrepancy between provability $\vdash_{\mathrm{QHM}}$ and validity $\models$, the following lemma will be useful. It enables us to make comparisons of the C-fragment of QHM with the predicate extension QS4(C) of uni-modal S4 with its modal operator C.

Lemma 3.5.(1): $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B) \supset (\mathrm{C}(A) \supset \mathrm{C}(B))$;

(2): $\vdash_{\mathrm{QHM}} \mathrm{C}(A) \supset A$;

(3): $\vdash_{\mathrm{QHM}} \mathrm{C}(A) \supset \mathrm{CC}(A)$;

(4): if $\vdash_{\mathrm{QHM}} A$, then $\vdash_{\mathrm{QHM}} \mathrm{C}(A)$;

(5): $\vdash_{\mathrm{QHM}} \forall x\mathrm{C}(A(x)) \supset \mathrm{C}(\forall x A(x))$.

Proof. We prove (1), (4) and (5).

(1): It suffices to prove $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B)\wedge\mathrm{C}(A) \supset \mathrm{C}(B)$. Since $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B)\wedge\mathrm{C}(A) \supset B$ and $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B)\wedge\mathrm{C}(A) \supset \mathrm{B}_i\,(\mathrm{C}(A \supset B) \wedge \mathrm{C}(A))$ for all $i \in N$ by CA, we have, by CI, $\vdash_{\mathrm{QHM}} \mathrm{C}(A \supset B)\wedge\mathrm{C}(A) \supset \mathrm{C}(B)$.

(4): Suppose $\vdash_{\mathrm{QHM}} A$. Then $\vdash_{\mathrm{QHM}} \mathrm{B}_i(A)$ for all $i \in N$ by Necessitation for B_i. Hence $\vdash_{\mathrm{QHM}} A \supset A\wedge\mathrm{B}_1(A) \wedge \cdots \wedge\mathrm{B}_n(A)$. By CI, $\vdash_{\mathrm{QHM}} A \supset \mathrm{C}(A)$. Hence $\vdash_{\mathrm{QHM}} \mathrm{C}(A)$.

(5): Since $\vdash_{\mathrm{QHM}} \mathrm{C}(A(a)) \supset A(a)\wedge\mathrm{B}_1\mathrm{C}(A(a))\wedge...\wedge\mathrm{B}_n\mathrm{C}(A(a))$ by CA, we have $\vdash_{\mathrm{QHM}} \forall x\mathrm{C}(A(x)) \supset \forall xA(x) \wedge \forall x\mathrm{B}_1\mathrm{C}(A(x)) \wedge ... \wedge \forall x\mathrm{B}_n\mathrm{C}(A(x))$. By $\forall$-B_i, we have $\vdash_{\mathrm{QHM}} \forall x\mathrm{C}(A(x)) \supset \forall xA(x)\wedge\mathrm{B}_1(\forall x\mathrm{C}(A(x))) \wedge ...\wedge\mathrm{B}_n(\forall x\mathrm{C}(A(x)))$. Regarding this as the upper formula of CI, we have $\vdash_{\mathrm{QHM}} \forall x\mathrm{C}(A(x)) \supset \mathrm{C}(\forall xA(x))$. ∎

Lemma 3.5.(5) is the Barcan formula for C with respect to $\forall$.

Let QS4(C) be the predicate extension of uni-modal S4 with the Barcan axiom with respect to $\forall$, i.e., it is the logic defined within $\mathcal{P}_{-\mathrm{B}}$ defined the axioms and inference rules of classical logics as well as the formulae (1)-(3), (5) and inference rule (4) of Lemma 3.5 for the modal operator C. The provability relation of QS4(C) is denoted by $\vdash_{\mathrm{QS4(C)}}$. Lemma 3.5 implies that for any $A \in \mathcal{P}_{-\mathrm{B}}$, if $\vdash_{\mathrm{QS4(C)}} A$, then $\vdash_{\mathrm{QHM}} A$. In fact, the converse holds, too, which will be discussed in Section 5.

4. Common Knowledge Logics CX, QCX and CY, QCY

As stated before, no finitary extensions of QKD4^n capture the semantic validity $\models$. In this section, we present extensions CX, CY and QCX, QCY of KD4^n and QKD4^n. These keep, respectively, the sets $\mathcal{P}_{-\text{Q}}$ and $\mathcal{P}$ of finitary formulae, but allow *infinitary* proofs. In the propositional case, both CX and CY turn to be equivalent to HM. In the predicate case, QCY is Kripke complete, and thus differs from QHM, but QCX is not known to be equivalent to QHM, QCY or neither.

4.1. Logics CX and QCX

To define CX and QCX, we adopt the formulae in Lemma 3.1 as an axiom schema and the inference rule corresponding to Lemma 3.4:

CA*: $\text{C}(A) \supset \text{B}_e(A)$, where $e \in N^*$;

$$\text{CI}_0^*: \ \frac{\{D \supset \text{B}_e(A) : e \in N^*\}}{D \supset \text{C}(A)}.$$

Although these two may look sufficient to determines $\text{C}(A)$ to be the common knowledge of A, the Kripke semantics has the Barcan property of B_i with respect to operator C, i.e., $\models \text{C}(A) \supset \text{B}_i\text{C}(A)$ for all $i \in N$. Therefore, we assume these Barcan formulae, too:

CB: $\text{C}(A) \supset \text{B}_i\text{C}(A)$ for all $i \in N$.

To have the balance between the syntactical system and Kripke semantics, we need this axiom CB. Note that Axiom CA has CB as part.

We define CX and QCX as follows:

CX: KD4^n + (CA*+ CI_0^*+ CB) within $\mathcal{P}_{-\text{Q}}$;

QCX: QKD4^n + (CA*+ CI_0^*+ CB) within $\mathcal{P}$.

Inference rule CI_0^* requires countable numbers of upper formulae. Accordingly, the definition of a finitary proof should be slightly modified into a countable tree where every path from the root is finite and a countably infinite branching occurs with inference CI_0^*. Infinitary proofs are allowed both in the propositional and predicate cases, but it will turn out that new provable formulae appear only in the predicate case.

Logic CX is deductively equivalent to the logic given in Kaneko [7], which is formulated as a sequent calculus.

Before stating the completeness result on CX, we mention the relationship of CX, QCX to HM, QHM.

Lemma 4.1.(1)(CA): $\vdash_{\text{QCX}} \text{C}(A) \supset A \wedge \text{B}_1\text{C}(A) \wedge ... \wedge \text{B}_n\text{C}(A)$;

(2)(CI): if $\vdash_{\text{QCX}} D \supset A \wedge \text{B}_1(D) \wedge ... \wedge \text{B}_n(D)$, then $\vdash_{\text{QCX}} D \supset \text{C}(A)$.

These claims hold with the replacement of QHM with HM.

Proof. (1) follows from CA* and CB.

(2): Suppose $\vdash_{\text{QCX}} D \supset A \wedge \text{B}_1(D) \wedge ... \wedge \text{B}_n(D)$. Then we can prove $\vdash_{\text{QCX}} D \supset \text{B}_e(A)$ for all $e \in N^*$. Therefore, by CI_0^*, $\vdash_{\text{QCX}} D \supset \text{C}(A)$. ■

We have the following theorem.

Theorem 4.2.(1):(Equivalence of HM and CX). For any $A \in \mathcal{P}_{-\text{Q}}$, $\vdash_{\text{HM}} A$ if and only if $\vdash_{\text{CX}} A$.

(2): For any $A \in \mathcal{P}$, if $\vdash_{\text{QHM}} A$, then $\vdash_{\text{QCX}} A$.

Proof. Lemma 4.1 implies (2) as well as the *only-if* part of (1). The *if* part of (1) follows from CA and 0_{HM} stated after Lemma 3.4. ■

The completeness of CX is a by-product of Theorem 4.2.(1) and Theorem 3.2. Nevertheless, since we do not have the completeness for QHM stated in Theorem 3.3, we cannot, at present, guarantee parallel results in the predicate case.

Remark: Two logics KD4^n + (CA*+ CI_0^*) within $\mathcal{P}_{-\text{Q}}$ and QKD4^n + (CA*+ CI_0^*) within $\mathcal{P}$ look natural. However, the following is known for the propositional case. Axiom CB is not provable in KD4^n + (CA*+ CI_0^*). This fact implies that KD4^n + (CA*+ CI_0^*) is Kripke incomplete. This result is obtained as follows: The Gentzen-style sequent formulation of KD4^n + (CA*+ CI_0^*) enjoys cut-elimination, which implies the full subformula property. Using this subformula property, we can prove that CB is not provable in KD4^n + (CA*+ CI_0^*). This method is not directly extended to the predicate case because of the Barcan axiom for B_i with respect to $\forall$ (it can be in the absence of the Barcan axiom).

4.2. Logics CY and QCY

Logic CX is complete as stated above, but we do not know whether or not its predicate extension QCX is complete. Nevertheless, we would obtain completeness if inference rule CI_0^* is strengthened into the following:

$$\text{CI}^*: \frac{\{D \supset T(\text{B}_e(A)) : e \in N^*\}}{D \supset T(\text{C}(A)).},$$

where $T(E)$ denotes any formulae of the following form:

$$\text{B}_{j_k}(D_k \supset ... \text{B}_{j_2}(D_2 \supset \text{B}_{j_1}(D_1 \supset E))...).$$

Note that $D_1, ..., D_k$ are any formulae in $\mathcal{P}$ and that $(j_k, ..., j_1)$ is any sequence in N^*. When $k = 0$, CI* becomes CI_0^*.

We define CY and QCY as follows:

CY: KD4^n + (CA*+CI*) within $\mathcal{P}_{-\text{Q}}$.

QCY: QKD4^n + (CA*+CI*) within $\mathcal{P}$.

Proofs in these logics are defined in the similar manner as in CX and QCX.

It can be verified that CY and QCY are extensions of CX and QCX, as follows. First, CI_0^* is a special case of CI*, as already stated.. Second, CB is also provable in QCY. Indeed, since $\vdash_{\text{QCX}} \text{C}(A) \supset \text{B}_i(\top \supset \text{B}_e(A))$ for all $e \in N^*$ by CA*, we have

$\vdash_{\mathrm{QCY}}$ C$(A) \supset$ B$_i(\top \supset$ C$(A))$, i.e., $\vdash_{\mathrm{QCY}}$ C$(A) \supset$ B$_i$C(A), where $\top$ denotes $\neg p \vee p$ for an atomic formula p.

Lemma 4.3.(1): For any $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{CX}} A$ implies $\vdash_{\mathrm{CY}} A$.

(2): For any $A \in \mathcal{P}$, $\vdash_{\mathrm{QCX}} A$ implies $\vdash_{\mathrm{QCY}} A$.

We have the following completeness theorem for CY and QCY.

Theorem 4.4 (Completeness for CY and QCY).

(1): For any formula $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{CY}} A$ if and only if $\vDash A$.

(2): For any formula $A \in \mathcal{P}$, $\vdash_{\mathrm{QCY}} A$ if and only if $\vDash A$.

Since Lemma 4.3.(1) states that CY is an extension of CX, and since CX is Kripke complete, we would obtain Theorem 4.4(1) if CI* is sound in any Kripke frame. This verification is straightforward. Hence, the completeness part of Theorem 4.4.(2) is crucial here. Tanaka [16] discusses this completeness of QCY as well as other possible variants. His proof is given under the language with no function symbols. Although function symbols are unavoidable for future applications, a proof of (2) for the language with function symbols can be obtained by modifying Tanaka's [16] proof.

Since $\{A \in \mathcal{P} : \vDash A\}$ is not recursively enumerable by Theorem 2.4, Theorem 4.4.(2) implies that the set $\{A \in \mathcal{P} : \vdash_{\mathrm{QCY}} A\}$ is also not recursively enumerable. This non-recursive-enumerability may be regarded as caused by admitting infinitary proofs. On the other hand, since $\{A \in \mathcal{P}_{-\mathrm{Q}} : \vdash_{\mathrm{CY}} A\}$ coincides with

$$\{A \in \mathcal{P}_{-\mathrm{Q}} : \ \vDash A\} = \{A \in \mathcal{P}_{-\mathrm{Q}} : \ \vdash_{\mathrm{HM}} A\} = \{A \in \mathcal{P}_{-\mathrm{Q}} : \ \vdash_{\mathrm{CX}} A\},$$

the set $\{A \in \mathcal{P}_{-\mathrm{Q}} : \vdash_{\mathrm{CY}} A\}$ remains to be recursively enumerable, even though we allow infinitary proofs in CY. Therefore, an infinitary proof is not solely a cause for non-recursive-enumerability.

Now, the jump occurring in QHM discussed in Section 3 is regarded as a gap from QKD4^n to QCY. There is no gap in the propositional case. Now, it is clearer what the conundrums discussed in Section 3 are.

It follows from Theorem 4.4 that QCY is a conservative extension of CY, which is explicitly stated as the following corollary.

Corollary 4.5. For any $A \in \mathcal{P}_{-\mathrm{Q}}$, $\vdash_{\mathrm{CY}} A$ if and only if $\vdash_{\mathrm{QCY}} A$.

We will discuss the relationship between QCY and QHM in the next section.

5. Comparisons of Common Knowledge Logics

In this section, we make comparisons of logics QKD4^n, QHM, QCX, QCY and QS4(C) as well as their propositional fragments. It will be known from these comparisons except with QCX that if one is an extension of another, the extension is conservative upon the other. In addition of this, we will get a good but still partial answer to the question of what the difference between QHM and QCY is. However, it will remain open whether QCX coincides with QHM or QCY (or with neither). We will obtain also the result that any formula which is valid but is not provable in

QHM contains a belief operator B_i, the common knowledge operator C as well as a quantifier.

First, we compare the provabilities of our logics for propositional formulae. Remark that we can add the semantical validity $\models A$ to the list of the provability statements in the following theorems.

Theorem 5.1 (Propositional Formulae). For any formula $A \in \mathcal{P}_{-\mathrm{Q}}$, the following six statements are all equivalent: (1) $\vdash_{\mathrm{HM}} A$; (1_{Q}) $\vdash_{\mathrm{QHM}} A$; (2) $\vdash_{\mathrm{CX}} A$; (2_{Q}): $\vdash_{\mathrm{QCX}} A$; and (3) $\vdash_{\mathrm{CY}} A$; (3_{Q}) $\vdash_{\mathrm{QCY}} A$.
If A is C-free, then we can add (0) $\vdash_{\mathrm{KD4}^n} A$ and (0_{Q}) $\vdash_{\mathrm{QKD4}^n} A$ to the above list.

Proof. By the definition of each logic, we have (1) $\Rightarrow$ (1_{Q}), (2_{Q}) $\Rightarrow$ (3_{Q}) and (2) $\Rightarrow$ (3). Theorem 4.2.(2) states (1_{Q}) $\Rightarrow$ (2_{Q}). Corollary 4.5 states (3) $\Leftrightarrow$ (3_{Q}). Theorem 4.2.(1) states (1) $\Leftrightarrow$ (2). These are described as follows:

$$\begin{array}{ccccc}
(1) \vdash_{\mathrm{HM}} A & \Leftrightarrow & (2) \vdash_{\mathrm{CX}} A & \Rightarrow & (3) \vdash_{\mathrm{CY}} A \\
\Downarrow & & \Downarrow & & \Updownarrow \\
(1_{\mathrm{Q}}) \vdash_{\mathrm{QHM}} A & \Rightarrow & (2_{\mathrm{Q}}) \vdash_{\mathrm{QCX}} A & \Rightarrow & (3_{\mathrm{Q}}) \vdash_{\mathrm{QCY}} A
\end{array}$$

Finally, we get (1) $\Leftrightarrow$ (3) from Theorems 3.2 and 4.4.(1). Thus, we have the equivalences of all the six claims. ■

It is an implication of Theorem 5.1 that QHM, QCX, QCY and their propositional fragments are all conservative extensions of $\mathrm{KD4}^n$.

Next, we compare the provabilities of these logics for C-free formulae.

Theorem 5.2 (C–Free Formulae). For any $A \in \mathcal{P}_{-\mathrm{C}}$, the following four are equivalent: (1) $\vdash_{\mathrm{QKD4}^n} A$; (2) $\vdash_{\mathrm{QHM}} A$; (3) $\vdash_{\mathrm{QCX}} A$; and (4) $\vdash_{\mathrm{QCY}} A$.

Proof. By definitions, we have (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4). By Theorem 4.2.(2), (2) $\Rightarrow$ (3) holds. Conversely, suppose (4). By Theorem 4.4.(2), we have $\models A$. Since A does not contain C, the semantic validity of A in QCY is equivalent to that in $\mathrm{QKD4}^n$. By the completeness for $\mathrm{QKD4}^n$ (Theorem 2.2.(2)), we have $\vdash_{\mathrm{QKD4}^n} A$. Thus, (1), (2), (3) and (4) are all equivalent. ■

One implication of Theorem 5.2 is that QHM, QCX and QCY are all conservative extensions of $\mathrm{QKD4}^n$.

Next, consider B-free formulae, i.e., ones in $\mathcal{P}_{-\mathrm{B}}$. That is, we consider the provabilities of formulae including no $\mathrm{B}_1, ..., \mathrm{B}_n$ in our predicate extensions of $\mathrm{KD4}^n$. In this consideration, we focus on the predicate extension QS4(C) of uni-modal S4, since comparisons with it give good hints to understand common knowledge (propositional and predicate) extensions of $\mathrm{KD4}^n$.

In the next lemma, ψ is the translator from $\mathcal{P}$ to $\mathcal{P}_{-\mathrm{B}}$ to associate, with each A, the formula $\psi(A)$ obtained from A by replacing all occurrences $\mathrm{B}_1, ..., \mathrm{B}_n$ in A by C. For example, $\psi(\mathrm{C}(A) \supset \mathrm{B}_{i_1} ... \mathrm{B}_{i_m}(A)) = \mathrm{C}(\psi A) \supset \mathrm{C} ... \mathrm{C}(\psi A)$. Note $\psi A = A$ for any $A \in \mathcal{P}_{-\mathrm{B}}$.

Lemma 5.3. For any $A \in \mathcal{P}$, if $\vdash_{\mathrm{QCY}} A$, then $\vdash_{\mathrm{QS4(C)}} \psi(A)$.

Proof. It suffices to prove that $\vdash_{\mathrm{QS4(C)}} \psi(D)$ for all axioms D for QCY, and that the inference rules translated by ψ from those for QCY are admissible in QS4(C). If

D is an instance of L1–L7, then $\vdash_{\mathrm{QS4}(C)} \psi(D)$. Let D be an instance of Axiom K, i.e., $\mathrm{B}_i(A \supset B) \supset (\mathrm{B}_i(A) \supset \mathrm{B}_i(B))$. Then $\psi(D) = \mathrm{C}(\psi A \supset \psi B) \supset (\mathrm{C}(\psi A) \supset \mathrm{C}(\psi B))$, which is an instance of an axiom in QS4(C). In the same manner, if D is an instance of D or 4, we have $\vdash_{\mathrm{QS4(C)}} \psi(D)$. Consider the Barcan $\forall x \mathrm{B}_i(A(x)) \supset \mathrm{B}_i(\forall x A(x))$. The translation is $\psi(\forall x \mathrm{B}_i(A(x)) \supset \mathrm{B}_i(\forall x A(x))) = \forall x \mathrm{C}(\psi A(x)) \supset \mathrm{C}(\forall x \psi A(x))$, which is an instance of the Barcan axiom for operator C. Consider an instance of CA* : $\mathrm{C}(A) \supset \mathrm{B}_e(A)$, where $e = (i_1, ..., i_m)$. Then $\psi(\mathrm{C}(A) \supset \mathrm{B}_e(A)) = \mathrm{C}(\psi A) \supset \mathrm{C}...\mathrm{C}(\psi A)$. This is provable in QS4(C).

Regarding the inference rules, here we consider only Necessitation and CI*.

Necessitaiton: Let $\vdash_{\mathrm{QS4(C)}} \psi(A)$. Then $\vdash_{\mathrm{QS4(C)}} \mathrm{C}(\psi A)$, which is equivalent to $\vdash_{\mathrm{QS4(C)}} \psi(\mathrm{B}_i(A))$.

CI*. Suppose $\vdash_{\mathrm{QS4(C)}} \psi(D \supset T(\mathrm{B}_e(A)))$ for all $e \in N^*$. Consider the specific one $\vdash_{\mathrm{QS4(C)}} \psi(D \supset T(\mathrm{B}_1(A)))$. Since $\psi(D \supset T(\mathrm{B}_1(A)))$ is $\psi(D) \supset \psi T(\mathrm{B}_1(A))$, and since $\psi T(\mathrm{B}_1(A)) = \psi(\mathrm{B}_{j_m}(D_m \supset ... \mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{B}_1(A))...)) = \mathrm{C}(\psi D_m \supset ... \mathrm{C}(\psi D_2 \supset \mathrm{C}(\psi D_1 \supset \mathrm{C}(\psi A))...) = \psi(\mathrm{B}_{j_m}(D_m \supset ... \mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{C}(A))...)) = \psi T(\mathrm{C}(A))$, we have $\vdash_{\mathrm{QS4(C)}} \psi(D \supset T(\mathrm{C}(A)))$. ■

In fact, the provabilities of QHM, QCX and QCY for B-free formulae collapse into that of the predicate extension QS4(C) of uni-modal S4, whose modality is C.

Theorem 5.4 (B-Free Formulae). For any formula $A \in \mathcal{P}_{-\mathrm{B}}$, the following four statements are all equivalent: (1): $\vdash_{\mathrm{QHM}} A$; (2): $\vdash_{\mathrm{QCX}} A$; (3): $\vdash_{\mathrm{QCY}} A$; and (4): $\vdash_{\mathrm{QS4(C)}} A$.

Proof. By Theorem 4.2.(2), we have (1) $\Rightarrow$ (2). By definition, (2) $\Rightarrow$ (3). By Lemma 5.3, we have (3) $\Rightarrow$ (4). Suppose (4). Then there is a proof of A in logic QS4(C). It suffices to show that the logical axioms and inference rules in QS4(C) are admissible in QHM. These are stated in Lemma 3.3. ■

In the propositional case, we have the parallel result to Theorem 5.4: for any formula $A \in \mathcal{P}_{-\mathrm{BQ}}$, ($1_{-\mathrm{Q}}$): $\vdash_{\mathrm{HM}} A$; ($2_{-\mathrm{Q}}$): $\vdash_{\mathrm{CX}} A$; ($3_{-\mathrm{Q}}$): $\vdash_{\mathrm{CY}} A$; and ($4_{-\mathrm{Q}}$): $\vdash_{\mathrm{S4(C)}} A$ as well as (1)–(4) of Theorem 5.4 are all equivalent.

Although logic QHM is incomplete, Theorems 5.1 and 5.4 imply that QHM is a conservative extension of HM and QKD4^n. Of course, QCY is a conservative extension of CY and QKD4^n, too. Therefore, the extension relation among these logics described in Diagram 1.1 keeps conservativity.

As state in the beginning of this section, we can add the semantic validity $\vDash A$ to the list in the above three theorems. Since these theorems imply that all the sets $\{A \in \mathcal{P}_{-\mathrm{Q}} : \vDash A\}$, $\{A \in \mathcal{P}_{-\mathrm{B}} : \vDash A\}$ and $\{A \in \mathcal{P}_{-\mathrm{C}} : \vDash A\}$ are all recursively enumerable, their union $\{A \in \mathcal{P}_{-\mathrm{Q}} \cup \mathcal{P}_{-\mathrm{B}} \cup \mathcal{P}_{-\mathrm{C}} : \vDash A\}$ is also recursively enumerable.

Let $\mathcal{P}_{\mathrm{QF}}$ be the set of all *quantifier-free* formulae in $\mathcal{P}$. We define the quantifier-free fragments $\mathrm{KD4}^n_{\mathrm{QF}}$, $\mathrm{HM}_{\mathrm{QF}}$, $\mathrm{CX}_{\mathrm{QF}}$ and $\mathrm{CY}_{\mathrm{QF}}$ in the same manners as KD4^n, HM, CX and CY by adopting $\mathcal{P}_{\mathrm{QF}}$ rather than $\mathcal{P}_{-\mathrm{Q}}$. Then all the theorems on these propositional fragments remain true for $\mathrm{KD4}^n_{\mathrm{QF}}$, $\mathrm{HM}_{\mathrm{QF}}$, $\mathrm{CX}_{\mathrm{QF}}$ and $\mathrm{CY}_{\mathrm{QF}}$. The last conclusion of the above remark becomes that $\{A \in \mathcal{P}_{\mathrm{QF}} \cup \mathcal{P}_{-\mathrm{B}} \cup \mathcal{P}_{-\mathrm{C}} : \vDash A\}$ is recursively enumerable. Note that $\vdash_{\mathrm{QHM}} A$ holds for any formula A in $\{A \in \mathcal{P}_{\mathrm{QF}} \cup \mathcal{P}_{-\mathrm{B}} \cup \mathcal{P}_{-\mathrm{C}} : \vDash A\}$. Hence, we have the following theorem.

Theorem 5.5 (Difference between QCY and QHM)(1): For any $A \in \mathcal{P}$, if

$\vdash_{\mathrm{QCY}} A$, *a fortioi,* $\models A$, but $\nvdash_{\mathrm{QHM}} A$, then A contains a belief operator B_i for at least one i, the common knowledge operator C and a quantifier.[3]

(2): The set $\{A \in \mathcal{P} : \vdash_{\mathrm{QCY}} A$ and $\nvdash_{\mathrm{QHM}} A\}$ is not recursively enumerable.

It is an important open problem to find a particular formula for (1). After all, such a formula includes B_i for some i, C as well as $\forall$ (or $\exists$). Conversely, for other formulae, the provability $\vdash_{\mathrm{QCY}}$ coincides with $\vdash_{\mathrm{QHM}}$. In the game theoretical applications in Kaneko-Nagashima [8] and [9], we have met formulae including C and quantifiers. Even formulae we can think of in game theoretical applications are not ones for (1).

We have been focussed on extensions of QHM rather than its fragments. On the other hand, Sturm-Wolter-Zakharyaschev [15] considered the monodic fragment of QHM under the *assumption* that the language has no m-ary function symbols for $m \geq 1$. We say that a formula A is *monodic* iff each of any subformula $\mathrm{B}_j(D)$ ($j \in N$) and $\mathrm{C}(D)$ of A contains at most one free variable. Without the assumption of no m-ary function symbols with $m \geq 1$, the formula obtained from a monadic formula by substitution with a term may not be monodic. They proved the Kripke completeness of the monodic fragment of QHM under the assumption of no m-ary function symbols with $m \geq 1$. This fragment is located between HM and QHM. Thus, the gap from QHM to QCY occurs after the monodic fragment of QHM.

6. Game Logic QGL_ω: Infinitary Approach

Since common knowledge is an infinitary concept, it would be a direct approach to formulate common knowledge in an infinitary extension of $\mathrm{KD4}^n$. Kaneko-Nagashima [8] and [9] took this approach and provided an infinitary epistemic logics GL_ω and QGL_ω, where common knowledge is explicitly formulated as an infinitary conjunctive formula. They developed these systems from the proof-theoretic point of view, which are now expected to be Kripke-complete from the result of Tanaka-Ono [19] and Tanaka [18]. In the propositional case, Kaneko [7] showed that logic HM is *faithfully* embedded into GL_ω (with a slight restriction).[4] In this section, we will give a connection from the logic QCY to QGL_ω. As in the previous sections, we will focus on the predicate case, but all the results in this section are obtained also in the propositional case.

First, we add new conjunction and disjunction symbols $\bigwedge$ and $\bigvee$ to the list of primitive symbols in Section 2.1. These are applied to infinite sets of formulae.

Let $\mathcal{Q}$ be a given set of formulae. We define $\mathcal{E}(\mathcal{Q})$ as follows:

IF1: $\mathcal{Q}' = \mathcal{Q} \cup \{(\bigwedge \Phi), (\bigvee \Phi) : \Phi$ is a countably infinite subset of $\mathcal{Q}$ containing at most a finite number of free variables$\}$;

IF2: $\mathcal{E}(\mathcal{Q})$ is the set of formulae defined from $\mathcal{Q}'$ by the standard finite induction, that is, (1): any expression in $\mathcal{Q}'$ belongs to $\mathcal{E}(\mathcal{Q})$; (2): if $A, B \in \mathcal{E}(\mathcal{Q})$, then $(\neg A), (A \wedge$

[3] Tanaka [16] showed that the occurrence of C in A must be positive, by applying the method of tree-sequent calculus.

[4] Heifetz [5] discussed the infinitary approach and fixed-point approach in the propositional case, by unifying these approaches into one system and proving its Kripke completeness.

$B), (A \vee B), (A \supset B)$ and $\mathrm{B}_1(A), ..., \mathrm{B}_n(A)$ belong to $\mathcal{E}(\mathcal{Q})$; and (3): if $A(a) \in \mathcal{E}(\mathcal{Q})$, then $\forall x A(x)$ and $\exists x A(x)$ belong to $\mathcal{E}(\mathcal{Q})$.

By replacing $\mathcal{Q}$ by $\mathcal{E}(\mathcal{Q})$, we define $\mathcal{E}^2(\mathcal{Q}) = \mathcal{E}(\mathcal{E}(\mathcal{Q}))$. In general, we define $\mathcal{E}^{m+1}(\mathcal{Q}) = \mathcal{E}(\mathcal{E}^m(\mathcal{Q}))$ for any nonnegative integer m. We adopt the set of formulae $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}}) := \bigcup_{m<\omega} \mathcal{E}^m(\mathcal{P}_{-\mathrm{C}})$, taking $\mathcal{P}_{-\mathrm{C}}$ as $\mathcal{Q}$. In the following, we call Φ an *allowable set* iff Φ is a countably infinite set of formulae in $\mathcal{E}^m(\mathcal{P}_{-\mathrm{C}})$ for some $m < \omega$ and contains a finite number of free variables.

In the above construction of the sets of formulae, we do not include common knowledge operator symbol C, since common knowledge is now expressed as an infinitary conjunctive formula in $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$. That is, the common knowledge of A is expressed as

$$\bigwedge\{\mathrm{B}_e(A) : e \in N^*\}, \tag{6.1}$$

which we denote by $\mathrm{C}_0(A)$. Note that syntactically, this is an abbreviation of (6.1) and differs from $\mathrm{C}(A)$.

In the finitary language, the conjunctive symbol $\wedge$ and disjunctive symbol $\wedge$ are applied to two formulae. Now, larger $\bigwedge$ and $\bigvee$ are applied to allowable sets Φ of formulae. Therefore, we add the modifications of the axioms L4, L5 and inference rules $\wedge$-Rule, $\vee$-Rule: for any allowable set Φ of formulae,

$\mathrm{L4}_\omega$: $\bigwedge \Phi \supset A$, where $A \in \Phi$;

$\mathrm{L5}_\omega$: $A \supset \bigvee \Phi$, where $A \in \Phi$;

and

$$\frac{\{A \supset B : B \in \Phi\}}{A \supset \bigwedge \Phi} \ (\bigwedge\text{-Rule}) \qquad \frac{\{A \supset B : A \in \Phi\}}{\bigvee \Phi \supset B} \ (\bigvee\text{-Rule}).$$

We denote the union of the axioms for $\mathrm{QKD4}^n$ and $\mathrm{L4}_\omega$, $\mathrm{L5}_\omega$, $\bigwedge$-Rule, $\bigvee$-Rule by $\mathrm{QKD4}^n_\omega$.

Since an allowable set Φ is infinite, we need also the Barcan property on each belief operator B_i with respect to $\bigwedge$: for any allowable set Φ,

$\bigwedge$-B: $\bigwedge \mathrm{B}_i(\Phi) \supset \mathrm{B}_i(\bigwedge \Phi)$,

where $\mathrm{B}_i(\Phi) := \{\mathrm{B}_i(A) : A \in \Phi\}$. In $\bigwedge \mathrm{B}_i(\Phi)$ and $\mathrm{B}_i(\bigwedge \Phi)$, player i believes every formula in Φ, and in the latter, he believes additionally the entirety of Φ. Hence, the latter is regarded as stronger than the former. Indeed, the converse is $\bigwedge$-B. To make direct comparisons with QCY, we need axiom $\bigwedge$-B.

We define GL_ω and QGL_ω by

GL_ω : $\mathrm{KD4}^n_\omega + \bigwedge$-B within $\mathcal{E}^\omega_{-\mathrm{Q}}(\mathcal{P}_{-\mathrm{CQ}})$;

QGL_ω : $\mathrm{QKD4}^n_\omega + \bigwedge$-B within $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$.

Recall that $\mathcal{E}^\omega_{-\mathrm{Q}}(\mathcal{P}_{-\mathrm{CQ}})$ is the propositional fragment of $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$. In the following, however, we focus only on QGL_ω.

A *proof* in QGL_ω is defined to be a countable tree in the same manner as in QCX. We write $\vdash_{\mathrm{Q}\omega} A$ iff there is a proof of A in QGL_ω.

First, we show that operator $\mathrm{C}_0(\cdot)$ in QGL_ω has the same properties as C in QCY.

Lemma 6.1.(1): $\vdash_{\mathrm{Q}\omega} \mathrm{C}_0(A) \supset \mathrm{B}_e(A)$ for all $e \in N^*$;

(2): If $\vdash_{Q\omega} D \supset T(\mathrm{B}_e(A))$ for all $e \in N^*$, then $\vdash_{Q\omega} D \supset T(\mathrm{C}_0(A))$, where $T(E)$ is any formula considered in inference CI.

Proof. (1) follows from the definition of $\mathrm{C}_0(A)$.

Consider (2). We prove by induction on the structure of T that $\vdash_{Q\omega} \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\} \supset T(\mathrm{C}_0(A))$. When T is the null symbol, the assertion holds.

Let $T(\mathrm{B}_e(A))$ be written as $\mathrm{B}_{j_m}(D_m \supset ...\mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{B}_e(A)))...)$. Suppose $\vdash_{Q\omega} \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\} \supset T(\mathrm{C}_0(A))$. Then $\vdash_{Q\omega} (D_{j_{m+1}} \supset \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\}) \supset (D_{j_{m+1}} \supset T(\mathrm{C}_0(A)))$. Hence $\vdash_{Q\omega} \mathrm{B}_{j_{m+1}}(D_{m+1} \supset \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\}) \supset \mathrm{B}_{j_{m+1}}(D_{j_{m+1}} \supset T(\mathrm{C}_0(A)))$. On the other hand, since $\vdash_{Q\omega} \bigwedge\{\mathrm{B}_{j_{m+1}}(D_{m+1} \supset T(\mathrm{B}_e(A)) : e \in N^*\} \supset \mathrm{B}_{j_{m+1}}(D_{m+1} \supset \bigwedge\{T(\mathrm{B}_e(A)) : e \in N^*\})$, we have $\vdash_{Q\omega} \bigwedge\{\mathrm{B}_{j_{m+1}}(D_{m+1} \supset T(\mathrm{B}_e(A))) : e \in N^*\} \supset \mathrm{B}_{j_{m+1}}(D_{m+1} \supset T(\mathrm{C}_0(A)))$. ■

The semantic valuation $(\mathcal{F}, \sigma, w) \models$ of Subsection 2.3 can be applied to any formula in $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$ just by modifying E3 and E4 into the following: for any allowable sets Φ :

E3$_\omega$: $(\mathcal{M}, \sigma, w) \models \bigwedge\Phi \iff (\mathcal{M}, \sigma, w) \models A$ for all $A \in \Phi$;

E4$_\omega$: $(\mathcal{M}, \sigma, w) \models \bigvee\Phi \iff (\mathcal{M}, \sigma, w) \models A$ for some $A \in \Phi$.

Then we have the following completeness, which is obtained by modifying the proof of the completeness result for QCY. (See the Appendix A, Remark A.12).

Theorem 6.2 (Completeness for QGL$_\omega$). For any $A \in \mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$, $\vdash_{Q\omega} A$ if and only if $\mathcal{M} \models A$ for all models $\mathcal{M}$.

In the comparison of infinitary QGL$_\omega$ with QCY, the following formulae in $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$ are essential. We call a formula A in $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$ a *cc-formula* iff (1) no infinitary disjunctions occur in A and (2) if $\bigwedge\Phi$ is a subformula of A, then $\bigwedge\Phi$ is expressed as $\mathrm{C}_0(B)$ for some B.

We will obtain a cc-formulae by translating a formula in $\mathcal{P}$ by interpreting $\mathrm{C}(\cdot)$ as $\mathrm{C}_0(\cdot)$. By this translation, $\mathcal{P}$ is embedded into $\mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$. We define $\psi_\mathrm{C} : \mathcal{P} \to \mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}})$ by

T0: $\psi_\mathrm{C}(A) = A$ for all atomic A;

T1: $\psi_\mathrm{C}(\neg A) = \neg\psi_\mathrm{C}(A)$;

T2: $\psi_\mathrm{C}(A \supset B) = \psi_\mathrm{C}(A) \supset \psi_\mathrm{C}(B)$;

T3: $\psi_\mathrm{C}(A \wedge B) = \psi_\mathrm{C}(A) \wedge \psi_\mathrm{C}(B)$; and $\psi_\mathrm{C}(A \vee B) = \psi_\mathrm{C}(A) \vee \psi_\mathrm{C}(B)$;

T4: $\psi_\mathrm{C}(\forall x A(x)) = \forall x \psi_\mathrm{C}(A(x))$; and $\psi_\mathrm{C}(\exists x A(x)) = \exists x \psi_\mathrm{C}(A(x))$;

T5: $\psi_\mathrm{C}(\mathrm{B}_i(A)) = \mathrm{B}_i(\psi_\mathrm{C}(A))$;

T6: $\psi_\mathrm{C}(\mathrm{C}(A)) = \bigwedge\{\mathrm{B}_e(\psi_\mathrm{C}(A)) : e \in N^*\}$ $(= \mathrm{C}_0(\psi_\mathrm{C}(A)))$.

It is easy to see that $\psi_\mathrm{C}(A)$ is a cc-formula for any $A \in \mathcal{P}$. We can prove also the following lemmas.

Lemma 6.3. ψ_C is a bijection from $\mathcal{P}$ to the set of all cc-formulae.

Lemma 6.4. For any $A \in \mathcal{P}$, $\models A$ if and only if $\models \psi_\mathrm{C}(A)$.

Translation ψ_C embeds logic QCY into QGL$_\omega$.

Theorem 6.5 (Faithful Embedding). For any $A \in \mathcal{P}$, $\vdash_{\mathrm{QCY}} A$ if and only if $\vdash_{\mathrm{Q}\omega} \psi_{\mathrm{C}}(A)$.

Proof. Suppose $\vdash_{\mathrm{QCY}} A$. Note that $\vdash_{\mathrm{Q}\omega} \psi_{\mathrm{C}}(B)$ for any instance B of the axioms for QCY: CA* and CI* are already verified in Lemma 6.1, and the classical inference rules translated by ψ_{C} are admissible in QGL_ω. Thus, a proof of A in QCY is translated into that of $\psi_{\mathrm{C}}(A)$ in QGL_ω. Therefore, $\vdash_{\mathrm{Q}\omega} \psi_{\mathrm{C}}(A)$.

Suppose $\vdash_{\mathrm{Q}\omega} \psi_{\mathrm{C}}(A)$. By Theorem 6.2, we have $\vDash \psi_{\mathrm{C}}(A)$. This is equivalent to $\vDash A$ by Lemma 6.4. Hence $\vdash_{\mathrm{QCY}} A$ by the completeness for QCY (Theorem 4.4). ■

In the propositional case, we have also the above faithful embedding theorem. It is an improvement of the embedding theorem obtained in Kaneko [7] in that no restriction on the $\wedge$-B axiom is needed here, while the $\wedge$-B axiom is restricted only to the cc-formulae in [7].

After all, the set $\{A \in \mathcal{E}^\omega(\mathcal{P}_{-\mathrm{C}}) : \vdash_{\mathrm{Q}\omega} A$ and A is a cc-formula$\}$ is not recursively enumerable, since $\{A \in \mathcal{P} : \vdash_{\mathrm{QCY}} A\}$ is not recursively enumerable by Wolter's result and completeness for QCY. This result looks natural in that QGL_ω is already an infinitary logic. Nevertheless, the same embedding holds in the propositional case, but the set $\{A \in \mathcal{E}^\omega_{-\mathrm{Q}}(\mathcal{P}_{-\mathrm{CQ}}) : \vdash_\omega A$ and A is a cc-formula$\}$ is recursively enumerable, where $\vdash_\omega$ is the provability relation of propositional GL_ω. Therefore, a conundrum still remains, but we know now that Wolter's non-recursive enumerability result is genuinely a problem in the predicate case.

The final remark is a larger infinitary logics than QGL_ω and GL_ω. Possible candidates are $\mathrm{QL}_{\omega_1\omega}(\mathrm{QKD4}^n)$ and $\mathrm{L}_{\omega_1\omega}(\mathrm{KD4}^n)$, which is more standard in the literature of infinitary logics (cf., Karp [10]). As far as we assume proper Barcan axioms, we would have the same embedding theorems. For larger sets of formulae create no further difficulties. In this sense, QGL_ω and GL_ω are the smallest choices of infinitary extensions so that all of QCY and CY (and HM) are faithfully embedded.

(*The following Appendices are included only in the discussion paper.*)

Appendix A: Proof of the Completeness for QCY

In this appendix, we prove the completeness theorems for logic QCY (Theorem 4.3), using an algebraic method along the line of Rasiowa-Sikorski [14]. The proof here is essentially the same as that given in Tanaka [16] except the treatment of function symbols here.

In the first subsection, we will provide some algebraic notions and two key lemmas, which are slightly modified from the Rasiowa-Sikorski lemma and a lemma given in Tanaka-Ono [19].

In the following, we will discuss Theorem 4.3 in the predicate case. In the propositional case, our proofs can be modified without difficulty.

The soundness part of Theorems 4.3 are standard, only noting the semantic validity relation $\vDash$ satisfies the inference rule AI* of QCY.

A.1 Some Algebraic Notions

Consider a Boolean algebra $(\mathbf{B}, \sqcap, \sqcup, \Leftrightarrow, \mathbf{0}, \mathbf{1})$. We define $a \leq b$ iff $a \sqcup b = b$, and write $a \rightarrow b$ for $\Leftrightarrow a \sqcup b$. Then $\leq$ is a lattice ordering on $\mathbf{B}$. We say that a nonempty subset F of $\mathbf{B}$ is a *filter* iff (1): $a \leq b$ and $a \in F$ imply $b \in F$ and (2): $a, b \in F$ implies $a \sqcap b \in F$. We say that a filter F is *prime* iff (1): $F \neq \mathbf{B}$ and (2): $a \sqcup b \in F$ implies $a \in F$ or $b \in F$. For any subset S of $\mathbf{B}$, the *greatest lower bound* of S in $(\mathbf{B}, \sqcap, \sqcup, \Leftrightarrow, \mathbf{0}, \mathbf{1})$ is denoted by $\sqcap S$, and the *least upper bound* of S is denoted by $\sqcup S$. Note that $\sqcap S$ and $\sqcup S$ may not exist, but if either exists, it is unique.

Let $(\mathcal{Q}_1, \mathcal{Q}_2)$ be a pair of countable sets of nonempty subsets of $\mathbf{B}$ so that $\sqcap Q_1$ and $\sqcup Q_2$ exist for all $Q_1 \in \mathcal{Q}_1$ and $Q_2 \in \mathcal{Q}_2$. We say that a prime filter F is a $(\mathcal{Q}_1, \mathcal{Q}_2)$*-filter* iff

(1): for any $Q_1 \in \mathcal{Q}_1$, $Q_1 \subseteq F$ implies $\sqcap Q_1 \in F$;

(2): for any $Q_2 \in \mathcal{Q}_2$, $\sqcup Q_2 \in F$ implies $Q_2 \cap F \neq \emptyset$.

The following is known as the Rasiowa-Sikorski lemma (cf., [14]).

Lemma A.1. Let $\mathbb{B}$ be a Boolean algebra, and $(\mathcal{Q}_1, \mathcal{Q}_2)$ a pair of countable sets of nonempty subsets of $\mathbf{B}$ satisfying that $\sqcap Q_1$ and $\sqcup Q_2$ exist for any $Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2$. For $a, b \in \mathbf{B}$, if $a \not\leq b$, then there is a $(\mathcal{Q}_1, \mathcal{Q}_2)$-filter F such that $a \in F$ and $b \notin F$.

We prepare one more concept. We say that $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, \Leftrightarrow, \mathbf{0}, \mathbf{1}, \Box_1, ..., \Box_n)$ is a *multi-modal algebra* iff

(1): $(\mathbf{B}, \sqcap, \sqcup, \Leftrightarrow, \mathbf{0}, \mathbf{1})$ is a Boolean algebra;

(2): for $i \in N$, $\Box_i$ is an operator on $\mathbf{B}$ satisfying the property that $\Box_i \mathbf{1} = \mathbf{1}$ and $\Box_i(a \sqcap b) = \Box_i a \sqcap \Box_i b$ for all $a, b \in \mathbf{B}$.

We denote the set of all $(\mathcal{Q}_1, \mathcal{Q}_2)$-filters of $\mathbb{B}$ by $\mathcal{F}_{(\mathcal{Q}_1,\mathcal{Q}_2)}(\mathbb{B})$. The following lemmas, A.2 and A.3, are Lemmas 2.7 and 2.8 of Tanaka-Ono [19]. Lemmas A.1 and A.3 will be used in the proof of the completeness.

Let H be a filter of a Boolean algebra $\mathbb{B}$. We define a binary relation $\sim_H$ on $\mathbf{B}$ by: $x \sim_H y \Longleftrightarrow$ both $x \rightarrow y$ and $y \rightarrow x$ belong to H. Then $\sim_H$ is a congruence relation on $\mathbf{B}$ with respect to $\sqcap, \sqcup$ and $\Leftrightarrow$. The quotient Boolean algebra $\mathbf{B}/\sim_H$ is denoted as $\mathbf{B}/H$. The equivalence class in $\mathbf{B}/H$ containing z is denoted by $|z|$. Moreover, for a subset Z of $\mathbf{B}$, the set $\{|z| : z \in Z\}$ is denoted by $|Z|$. Note that $|x| \leq |y| \Longleftrightarrow x \rightarrow y \in H$.

Let $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, \Leftrightarrow, \mathbf{0}, \mathbf{1}, \Box_1, ..., \Box_n)$ be a multi-modal algebra. Then we define $\Box_i^{-1} F = \{x \in \mathbf{B} : \Box_i x \in F\}$ for any $F \subseteq \mathbf{B}$. If F is a filter, so is $\Box_i^{-1} F$.

Lemma A.2. Let $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, \Leftrightarrow, \mathbf{0}, \mathbf{1}, \Box_1, ..., \Box_n)$ be a multi-modal algebra. Let $(\mathcal{Q}_1, \mathcal{Q}_2)$ be a fixed pair of countable sets of nonempty subsets of $\mathbf{B}$ satisfying that $\sqcap Q_1$ and $\sqcup Q_2$ exist for any $Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2$. Suppose the following conditions: for all $i \in N$,

(1): for all $Q_1 \in \mathcal{Q}_1$, $\sqcap \Box_i Q_1 := \sqcap\{\Box_i a : a \in Q_1\}$ exists and $\sqcap \Box_i Q_1 = \Box_i \sqcap Q_1$;

(2): $\{\Box_i(a \rightarrow b) : b \in Q_1\} \in \mathcal{Q}_1$ for all $a \in \mathbf{B}$ and all $Q_1 \in \mathcal{Q}_1$;

(3): $\{\Box_i(b \rightarrow a) : b \in Q_2\} \in \mathcal{Q}_1$ for all $a \in \mathbf{B}$ and all $Q_2 \in \mathcal{Q}_2$.

Then, for any $(\mathcal{Q}_1, \mathcal{Q}_2)$-filter F of $\mathbb{B}$, the Boolean algebra $\mathbf{B}/\Box_i^{-1} F$ satisfies

(2.1): $\sqcap |Q_1|$ exists and $\sqcap |Q_1| = |\sqcap Q_1|$ for all $Q_1 \in \mathcal{Q}_1$;

(2.2): $\sqcup |Q_2|$ exists and $\sqcup |Q_2| = |\sqcup Q_2|$ for all $Q_2 \in \mathcal{Q}_2$.

Lemma A.3. Let $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, \Leftrightarrow, \mathbf{0}, \mathbf{1}, \Box_1, ..., \Box_n)$ be a multi-modal algebra. Let $(\mathcal{Q}_1, \mathcal{Q}_2)$ be a fixed pair of countable sets of nonempty subsets of $\mathbf{B}$ satisfying that $\sqcap Q_1$ and $\sqcup Q_2$ exist for any $Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2$. Suppose that $(\mathcal{Q}_1, \mathcal{Q}_2)$ satisfies the conditions (1),(2) and (3) of Lemma A.2*. Then, for any $i \in N$ and $F \in \mathcal{F}_{(\mathcal{Q}_1,\mathcal{Q}_2)}(\mathbb{B})$, if $\Box_i a \notin F$, there exists a $G \in \mathcal{F}_{(\mathcal{Q}_1,\mathcal{Q}_2)}(\mathbb{B})$ such that $\Box_i^{-1} F \subseteq G$ and $a \notin G$.

A.2 Lindenbaum Algebra

We add a countably infinite number of constant symbols $\mathbf{d}_0, \mathbf{d}_1, ...$ We denote the set of formulae for the extended language by $\mathcal{P}^*$, and the set of closed formulae by $\mathcal{CP}^*$. We denote the provability relation of QCY with $\mathcal{CP}^*$ by $\vdash^*_{\mathrm{QCY}}$. For any $A, B \in \mathcal{CP}^*$, we define $A \sim B$ iff $\vdash^*_{\mathrm{QCY}} (A \supset B) \wedge (B \supset A)$. This relation $\sim$ is an equivalence relation over $\mathcal{CP}^*$. Therefore, we have the quotient set $\mathcal{CP}^*/\sim$. We denote the equivalence class in $\mathcal{CP}^*/\sim$ including A by $[\![A]\!]$.

In $\mathcal{CP}^*/\sim$, we define elements $\mathbf{0}, \mathbf{1}$ and operations $\sqcap, \sqcup, \Leftrightarrow, \Box_1, ..., \Box_n$ by

(1): $\mathbf{0} = [\![\neg A \wedge A]\!]$ and $\mathbf{1} = [\![\neg A \vee A]\!]$;

(2): $[\![A]\!] \sqcap [\![B]\!] = [\![A \wedge B]\!]$, $[\![A]\!] \sqcup [\![B]\!] = [\![A \vee B]\!]$, and $\Leftrightarrow [\![A]\!] = [\![\neg A]\!]$;

(3): $\Box_i [\![A]\!] = [\![\mathrm{B}_i(A)]\!]$ for all $i = 1, ..., n$.

Then we have the following lemma.

Lemma A.4. $\mathbb{L} = (\mathcal{CP}^*/\sim, \mathbf{0}, \mathbf{1}, \sqcap, \sqcup, \Leftrightarrow, \Box_1, ..., \Box_n)$ is a multi-modal algebra.

Proof. It is standard to show that $(\mathcal{CP}^*/\sim, \mathbf{0}, \mathbf{1}, \sqcap, \sqcup, \Leftrightarrow)$ is a Boolean algebra. Let $i \in N$. Since $\vdash^*_{\mathrm{QCY}} (\neg A \vee A \supset \mathrm{B}_i(\neg A \vee A)) \wedge (\mathrm{B}_i(\neg A \vee A) \supset \neg A \vee A))$, we have $\Box_i \mathbf{1} = \mathbf{1}$. Since $\vdash^*_{\mathrm{QCY}} (\mathrm{B}_i(A \wedge B) \supset \mathrm{B}_i(A) \wedge \mathrm{B}_i(B)) \wedge (\mathrm{B}_i(A) \wedge \mathrm{B}_i(B) \supset \mathrm{B}_i(A \wedge B))$, we have $\Box_i([\![A]\!] \sqcap [\![B]\!]) = \Box_i [\![A]\!] \sqcap \Box_i [\![B]\!]$. ■

In the following, we call $\mathbb{L}$ the *Lindenbaum algebra*. In the main step of the completeness theorem, we will use Lemmas A.1 and A.3. For this purpose, we should prove the following lemma, which will guarantee the condition of Lemma A.2.

Lemma A.5. Let $T(\mathrm{B}_e(A))$ be formula $\mathrm{B}_{j_m}(D_m \supset \mathrm{B}_{j_{m-1}}(D_{m-1} \supset ... \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{B}_e(A))...))$ for $e \in N^*$, and $T(\mathrm{C}(A))$ formula $\mathrm{B}_{j_m}(D_m \supset \mathrm{B}_{j_{m-1}}(D_{m-1} \supset ... \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{C}(A))...))$. Then, in the Lindenbaum algebra $\mathbb{L}$,

(1): $\sqcap\{[\![T(\mathrm{B}_e(A))]\!] : e \in N^*\} = [\![T(\mathrm{C}(A))]\!]$;

(2): for any $i \in N$, $\sqcap\{\Box_i [\![T(\mathrm{B}_e(A))]\!] : e \in N^*\} = \Box_i [\![T(\mathrm{C}(A))]\!]$.

Proof. Consider (1). First, let us see that $[\![T(\mathrm{C}(A))]\!]$ is a lower bound of $\{[\![T(\mathrm{B}_e(A))]\!] : e \in N^*\}$. This follows from the fact that $\vdash^*_{\mathrm{QCY}} T(\mathrm{C}(A)) \supset T(\mathrm{B}_e(A))$ for all $e \in N^*$. Now, let $[\![D]\!]$ be a lower bound of $\{[\![T(\mathrm{B}_e(A))]\!] : e \in N^*\}$. This means that $\vdash^*_{\mathrm{QCY}} D \supset T(\mathrm{B}_e(A))$ for all $e \in N^*$. Hence, by CI*, we have $\vdash^*_{\mathrm{QCY}} D \supset T(\mathrm{C}(A))$. This means that $[\![T(\mathrm{C}(A))]\!]$ is greater than or equal to D in $\mathbb{L}$. Thus, $[\![T(\mathrm{C}(A))]\!]$ is the greatest lower bound of $\{[\![T(\mathrm{B}_e(A))]\!] : e \in N^*\}$.

(2) follows from (1) by considering $\mathrm{B}_i(\neg A \vee A \supset T(\mathrm{B}_e(A)))$ in (1) instead of $T(\mathrm{B}_e(A))$. ■

Now we define a particular $(\mathcal{Q}_1, \mathcal{Q}_2)$ as follows. First, define $\mathcal{S}_0, \mathcal{S}_1, \ldots$ by the following induction:

(1): $\mathcal{S}_0 = \{\{[\![\mathrm{B}_e(D)]\!] : e \in N^*\} : D \in \mathcal{CP}^*\}$;

(2): $\mathcal{S}_{m+1} = \{\{\Box_i(b \to a) : a \in S\} : b \in \mathcal{CP}^*/\sim,\ S \in \mathcal{S}_m \text{ and } i \in N\}$.

Next, for each $C \in \mathcal{CP}^*$ of the form $\forall x D(x)$ or $\exists x D(x)$, define $S_C = \{[\![D(t)]\!] : t$ is a closed term$\}$, where $D(t)$ is obtained from expression $D(x)$ by substituting t for all occurrences of x in $D(x)$. We define

(3): $\mathcal{T} = \{S_C : C \in \mathcal{CP}^*$ and is of the form $\forall x D(x)$ or $\exists x D(x)\}$.

Then we define $(\mathcal{Q}_1, \mathcal{Q}_2) = (\bigcup_{m \in \omega} \mathcal{S}_m \cup \mathcal{T}, \mathcal{T})$. Then $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are countable sets of sets of closed formulae.

Lemma A.6.(1): For any $Q_1 \in \mathcal{Q}_1$ and $Q_2 \in \mathcal{Q}_2$, $\sqcap Q_1$ and $\sqcup Q_2$ exist;

(2): $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ satisfies the conditions (1), (2), (3) of Lemma A.2.

Proof.(1): Consider $S \in \mathcal{S}_m$. This S is written as $\{[\![\mathrm{B}_{j_m}(D_m \supset \ldots \mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{B}_e(A)))\ldots)]\!] : e \in N^*\}$. Hence $\sqcap S$ is $[\![\mathrm{B}_{j_m}(D_m \supset \ldots \mathrm{B}_{j_2}(D_2 \supset \mathrm{B}_{j_1}(D_1 \supset \mathrm{C}(A)))\ldots)]\!]$ by Lemma A.5.

Consider S_C, where $C \in \mathcal{CP}^*$ and is of the form $\forall x D(x)$ or $\exists x D(x)$. By L6 and L7, we have $\sqcap S_C = [\![\forall x D(x)]\!]$ and $\sqcup S_C = [\![\exists x D(x)]\!]$.

(2): Consider the condition (1) of Lemma A.2. Let $S \in \mathcal{S}_m$. In the same as (1), we have $\sqcap \Box_i S = \Box_i(\sqcap S)$ by Lemma A.5.(2). Hence its existence is also proved.

Let $S_C \in \mathcal{T}$. Then $\Box_i(\sqcap S_C) = \Box_i[\![\forall x D(x)]\!] = [\![\mathrm{B}_i(\forall x D(x))]\!] = [\![\forall x \mathrm{B}_i(D(x))]\!] = \sqcap\{[\![\mathrm{B}_i(D(t))]\!] : t$ is a closed term$\} = \sqcap \Box_i S_C$. Hence its existence is also proved.

Consider (2) of Lemma A.2. Let $Q_1 \in \mathcal{Q}_1$ and $a \in \mathcal{CP}^*/\sim$. Then if $Q_1 \in \mathcal{S}_m$ for some m, then $\{\Box_i(a \to b) : b \in Q_1\} \in \mathcal{S}_{m+1} \subseteq \mathcal{Q}_1$. Suppose $Q_1 \in \mathcal{T}$. Then $Q_1 = \{D(t) : t$ is a closed term$\}$ for some closed $\forall x D(x)$. Let $a = [\![A]\!]$. Then $\{\Box_i(a \to b) : b \in Q_1\} = \{\Box_i(\Leftrightarrow a \sqcup b) : b \in Q_1\} = \{\Box_i([\![\neg A]\!] \sqcup [\![D(t)]\!]) : t$ is a closed term$\} = \{\Box_i([\![\neg A \vee D(t)]\!]) : t$ is a closed term$\} \in \mathcal{T} \subseteq \mathcal{Q}_1$.

Consider (3) of Lemma A.2. Let $Q_2 \in \mathcal{Q}_2 = \mathcal{T}$ and $a \in \mathbf{B}$. Then $Q_2 = \{D(t) : t$ is a closed term$\}$ for some closed $\forall x D(x)$. Let $a = [\![A]\!]$. Then $\{\Box_i(b \to a) : b \in Q_2\} = \{\Box_i(\Leftrightarrow b \sqcup a) : b \in Q_2\} = \{\Box_i([\![\neg D(t)]\!] \sqcup [\![A]\!]) : t$ is a closed term$\} = \{\Box_i([\![\neg D(t) \vee A]\!]) : t$ is a closed term$\} = \{([\![\mathrm{B}_i(\neg D(t) \vee A)]\!]) : t$ is a closed term$\} \in \mathcal{T} \subseteq \mathcal{Q}_1$. ■

Now we are going to define a Kripke frame $\mathcal{F} = (W; R_1, \ldots, R_n; M)$, and an interpretation $\mathcal{I}^d(w) = [\tilde{d}_0, \tilde{d}_1, \ldots, \tilde{f}_0, \tilde{f}_1, \ldots; \tilde{P}_0^w, \tilde{P}_2^w, \ldots]$ in $\mathcal{F}$ as follows:

(1): $W = \mathcal{F}_{(\mathcal{Q}_1, \mathcal{Q}_2)}(\mathbb{L})$;

(2): for all $i \in N$, wR_iu if and only if $\Box_i^{-1} w \subseteq u$;

(3): M is the set of all closed terms;

(4): for newly introduced constant symbol $\mathbf{d}_k$, its interpretation $\tilde{d}_k = \mathbf{d}_k$;

(5): for any m-ary function symbol $\mathbf{f}_k$, $\tilde{f}_k(t_1, \ldots, t_m) = \mathbf{f}_k(t_1, \ldots, t_m)$;

(6): for any m-ary predicate symbol $\mathbf{P}_k$ and any $w \in W$, $\tilde{P}_k^w(t_1, ..., t_m) = \top$ if and only if $[\![\mathbf{P}_k(t_1, ..., t_m)]\!] \in w$.

We will prove $\mathcal{M}^d = (\mathcal{F}, \mathcal{I}^d) = ((W; R_1, ..., R_n; M), \mathcal{I}^d)$ is a Kripke model for the extended $L^d = [\mathbf{d}_0, \mathbf{d}_1, ..., \mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, ...]$. Let $\mathcal{I}$ be the interpretation obtained from $\mathcal{I}^d$ by deleting the interpretations of $\mathbf{d}_k$ for $k = 0, ...$ We will prove that $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ is a Kripke model for the original language $L = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, ...]$.

Lemma A.7. Each R_i is serial and transitive.

Proof. Consider seriality. Let $w \in W$. Consider $\Box_i[\![\neg A \wedge A]\!] = [\![\mathrm{B}_i(\neg A \wedge A)]\!]$. By Axiom D, we have $\Box_i[\![\neg A \wedge A)]\!] = [\![\neg A \wedge A]\!] = \mathbf{0}$. Since w is a prime filter, we have $\mathbf{0} \notin w$. By Lemma A.3, we have $u \in \mathcal{F}_{(\mathcal{Q}_1, \mathcal{Q}_2)}(\mathbb{L})$ such that $\Box_i^{-1} w \subseteq u$, i.e., wR_iu (and $\mathbf{0} \notin u$).

Consider transitivity. Suppose wR_iu and uR_iv. Then $\Box_i^{-1}w \subseteq u$ and $\Box_i^{-1}u \subseteq v$. Let $[\![A]\!] \in \Box_i^{-1}w$. Then $[\![\mathrm{B}_i(A)]\!] = \Box_i[\![A]\!] \in w$. By Axiom 4, we have $\Box_i[\![A]\!] = [\![\mathrm{B}_i(A)]\!] \leq [\![\mathrm{B}_i\mathrm{B}_i(A)]\!] = \Box_i\Box_i[\![A]\!]$. Since w is a filter, we have $\Box_i\Box_i[\![A]\!] \in w$. This implies $\Box_i[\![A]\!] \in \Box_i^{-1}w \subseteq u$. Hence $[\![A]\!] \in \Box_i^{-1}u \subseteq v$. ■

Let σ be an assignment from $\{\mathbf{a}_0, \mathbf{a}_1, ...\}$ to M. For any term t, we define t^σ to be the term obtained from t by replacing each free variable $\mathbf{a}_k$ occurring in t by closed term $\sigma(\mathbf{a}_k)$, and for $A \in \mathcal{P}^*$, we define A^σ to be the formula obtained from A by replacing each free variable $\mathbf{a}_k$ occurring in A by $\sigma(\mathbf{a}_k)$. The following lemma can be proved by induction on the length of a term.

Lemma A.8. For any term t, $V(t, (\mathcal{I}^d, \sigma)) = t^\sigma$.

Lemma A.9. For any $A \in \mathcal{P}^*$ and $w \in W$, $(\mathcal{M}^d, \sigma, w) \models A$ if and only if $[\![A^\sigma]\!] \in w$.

Proof. Consider an atomic formula $\mathbf{P}_k(t_1, ..., t_m)$. Then $(\mathcal{M}^d, \sigma, w) \models \mathbf{P}_k(t_1, ..., t_m)$ $\Leftrightarrow \tilde{P}_k^w(V(t_1, (\mathcal{I}^d, \sigma)), ..., V(t_m, (\mathcal{I}^d, \sigma))) = \top \Leftrightarrow \tilde{P}_k^w(t_1^\sigma, ..., t_m^\sigma) = \top \Leftrightarrow [\![\mathbf{P}_k(t_1^\sigma, ..., t_m^\sigma)]\!] \in w \Leftrightarrow [\![\mathbf{P}_k(t_1, ..., t_m)^\sigma]\!] \in w$.

Now, consider a nonatomic formula A in $\mathcal{P}^*$. We suppose the induction hypothesis that the assertion holds for any closed subformula of A. We should consider the eight cases based on the outermost connectives: $\neg, \supset, \wedge, \vee, \forall, \exists$, B_1, ...,B_n and C. We consider only $\neg, \forall$, B_i and C.

$(\neg)$: Suppose $(\mathcal{M}^d, \sigma, w) \models \neg A$. Then $(\mathcal{M}^d, \sigma, w) \not\models A$. By the induction hypothesis, we have $[\![A^\sigma]\!] \notin w$. Since $[\![\neg A^\sigma]\!] \sqcup [\![A^\sigma]\!] = [\![\neg A^\sigma \vee A^\sigma]\!] = \mathbf{1} \in w$, and w is a prime filter, we have $[\![\neg A^\sigma]\!] \in w$.

Suppose $[\![\neg A^\sigma]\!] \in w$. Since w is a prime filter, we have $[\![A^\sigma]\!] \notin w$, which implies Suppose $(\mathcal{M}^d, \sigma, w) \not\models A$ by the induction hypothesis. Thus, $(\mathcal{M}^d, \sigma, w) \models \neg A$.

$(\supset)$: Suppose $(\mathcal{M}^d, \sigma, w) \models A \supset B$. Then $(\mathcal{M}^d, \sigma, w) \not\models A$ or $(\mathcal{M}^d, \sigma, w) \models B$. By the induction hypothesis, we have $[\![A^\sigma]\!] \notin w$ or $[\![B^\sigma]\!] \in w$. Since $[\![\neg A^\sigma]\!] \in w$ or $[\![B^\sigma]\!] \in w$, and since $[\![\neg A^\sigma]\!] \leq [\![A^\sigma \supset B^\sigma]\!]$ and $[\![B^\sigma]\!] \leq [\![A^\sigma \supset B^\sigma]\!]$, we have $[\![A^\sigma \supset B^\sigma]\!] \in w$.

Suppose $[\![A^\sigma \supset B^\sigma]\!] \in w$. Then $[\![\neg A^\sigma \vee B^\sigma]\!] = [\![\neg A^\sigma]\!] \sqcup [\![B^\sigma]\!] \in w$. Since w is a prime filter, we have $[\![\neg A^\sigma]\!] \in w$ or $[\![B^\sigma]\!] \in w$. Hence $[\![A^\sigma]\!] \notin w$ or $[\![B^\sigma]\!] \in w$. By the induction hypothesis, we have $(\mathcal{M}^d, \sigma, w) \not\models A$ or $(\mathcal{M}^d, \sigma, w) \models B$. Thus, $(\mathcal{M}^d, \sigma, w) \models A \supset B$.

$(\forall)$: Suppose $(\mathcal{M}^d, \sigma, w) \models \forall x A(x)$. Then $(\mathcal{M}^d, \sigma', w) \models A(\mathbf{a})$ for any $\sigma' \underset{\mathbf{a}}{=} \sigma$. By the induction hypothesis, we have $[\![A(\mathbf{a})^{\sigma'}]\!] \in w$ for all $\sigma' \underset{\mathbf{a}}{=} \sigma$. This means $[\![A(t)^{\sigma}]\!] \in w$ for all closed t. The greatest lower bound of $\{[\![A(t)]\!] : t$ is a closed term$\}$ is $[\![\forall x A(x)^{\sigma}]\!] = [\![(\forall x A(x))^{\sigma}]\!]$ and belongs to w, since w is a $(\mathcal{Q}_1, \mathcal{Q}_2)$-filter.

Suppose $[\![\forall x A(x)^{\sigma}]\!] \in w$. Then $[\![\forall x A(x)^{\sigma}]\!] \leq [\![A(t)]\!]$ for any closed term t. Hence $[\![A(t)]\!] \in w$ for all closed term t. Hence $(\mathcal{F}, I^d, \sigma, w) \models A(t)$ for all closed terms t. Let $\sigma' \underset{\mathbf{a}}{=} \sigma$ and $\sigma'(\mathbf{a}) = t \in M$. Then $(\mathcal{M}^d, \sigma', w) \models A(\mathbf{a})$. This means $(\mathcal{M}^d, \sigma', w) \models A(\mathbf{a})$ for any $\sigma' \underset{\mathbf{a}}{=} \sigma$, which implies $(\mathcal{M}^d, \sigma, w) \models \forall x A(x)$.

$(\exists)$: Suppose $(\mathcal{M}^d, \sigma, w) \models \exists x A(x)$. Then $((\mathcal{F}, \mathcal{I}^d), \sigma', w) \models A(\mathbf{a})$ for some $\sigma' \underset{\mathbf{a}}{=} \sigma$. Let $s = \sigma'(\mathbf{a})$. Then $(\mathcal{M}^d, \sigma', w) \models A(\mathbf{a})$. By the induction hypothesis, we have $[\![A(\mathbf{a})^{\sigma'}]\!] \in w$. Since $[\![A(\mathbf{a})^{\sigma'}]\!] \leq [\![\exists x A(x)^{\sigma'}]\!] = [\![\exists x A(x)^{\sigma}]\!]$, we have $[\![\exists x A(x)^{\sigma}]\!] \in w$ since w is a filter.

Suppose $[\![\exists x A(x)^{\sigma}]\!] \in w$. Since w is a $(\mathcal{Q}_1, \mathcal{Q}_2)$-filter, we have $\{A(t)^{\sigma} : t$ is a closed term$\} \cap w \neq \emptyset$. Let $[\![A(s)^{\sigma}]\!] \in w$. By the induction hypothesis, we have $(\mathcal{M}^d, \sigma, w) \models A(s)$. Taking $\sigma'(\mathbf{a}) = s$, we have $(\mathcal{M}^d, \sigma', w) \models A(\mathbf{a})$, which implies $(\mathcal{M}^d, \sigma, w) \models \exists x A(x)$.

(B_i) : Suppose $(\mathcal{M}^d, \sigma, w) \models \mathrm{B}_i(A)$. Then $(\mathcal{M}^d, \sigma, u) \models A$ for any $u \in R_i(w) := \{v \in W : wR_iv\}$. By the induction hypothesis, $[\![A^{\sigma}]\!] \in u$ for any $u \in R_i(w)$. Suppose $\Box_i[\![A^{\sigma}]\!] \notin w$. Then, by Lemma A.3, we have another $u \in \mathcal{F}_{(\mathcal{Q}_1,\mathcal{Q}_2)}(\mathbb{L})$ such that $\Box_i^{-1}w \subseteq u$ and $[\![A^{\sigma}]\!] \notin u$. This is a contradiction.. Hence $[\![\mathrm{B}_i(A)^{\sigma}]\!] = \Box_i[\![A^{\sigma}]\!] \in w$.

Suppose $[\![\mathrm{B}_i(A)^{\sigma}]\!] = \Box_i[\![A^{\sigma}]\!] \in w$. Then for any u with $\Box_i^{-1}w \subseteq u$, $[\![A^{\sigma}]\!] \in u$. By the induction hypothesis, we have $(\mathcal{M}^d, \sigma, u) \models A$ for all $u \in R_i(w)$.

(C) : Suppose $(\mathcal{M}^d, \sigma, w) \models \mathrm{C}(A)$. Then $(\mathcal{M}^d, \sigma, u) \models A$ for all u reachable from w. By the induction hypothesis, we have $[\![A]\!] \in u$ for all u reachable from w. Let e be any element in N^*. Now, suppose $[\![\mathrm{B}_e(A)]\!] \in u$ for all u reachable from w. Let u be any reachable world from w. If $\Box_i[\![\mathrm{B}_e(A)]\!] \notin u$, then, by Lemma A.3, we have another $v \in \mathcal{F}_{(\mathcal{Q}_1,\mathcal{Q}_2)}(\mathbb{L})$ such that $\Box_i^{-1}u \subseteq v$ and $[\![\mathrm{B}_e(A)]\!] \notin v$, which is a contradiction. Hence $\Box_i[\![\mathrm{B}_e(A^{\sigma})]\!] \in u$. That is, $[\![\mathrm{B}_i\mathrm{B}_e(A^{\sigma})]\!] \in u$ for all $i \in N$ and all u reachable from w. Hence we proved that $[\![\mathrm{B}_e(A^{\sigma})]\!] \in u$ for all $e \in N^*$ and u reachable from w. Thus, $[\![\mathrm{B}_e(A^{\sigma})]\!] \in w$ for all $e \in N^*$. The greatest lower bound $\sqcap\{[\![\mathrm{B}_e(A^{\sigma})]\!] : e \in N^*\}$ is $[\![\mathrm{C}(A^{\sigma})]\!]$, and belongs to w by Lemma A.5 and the definition of $(\mathcal{Q}_1, \mathcal{Q}_2)$, i.e., $[\![\mathrm{C}(A^{\sigma})]\!] \in w$.

Suppose $[\![\mathrm{C}(A^{\sigma})]\!] \in w$. Since $[\![\mathrm{C}(A^{\sigma})]\!] \leq [\![\mathrm{B}_e(A^{\sigma})]\!]$ for all $e \in N^*$, we have $[\![\mathrm{B}_e(A^{\sigma})]\!] \in w$ for all $e \in N^*$. It follows from this fact that $[\![A^{\sigma}]\!] \in u$ for all u reachable from w. Hence $(\mathcal{M}^d, \sigma, u) \models A$ for all u reachable from w. Hence $(\mathcal{M}^d, \sigma, w) \models \mathrm{C}(A)$. ■

Lemma A.10. For any closed $A \in \mathcal{P}$, $\vdash^*_{\mathrm{QCY}} A$ if and only if $\vdash_{\mathrm{QCY}} A$.

Proof. We show the *only-if* part. Let $\vdash^*_{\mathrm{QCY}} A$. Then there is a proof P of A in QCY in $\mathcal{CP}^*$. Then P may have a countable number newly introduced constant symbols $d_0, d_1, ...$, and P has a countable free variables. Let P' be the tree obtained from P by substituting simultaneously $\mathbf{a}_{2m+1}$ for all the occurrences of free variables $\mathbf{a}_m$ in P for all $m = 0, 1, ...$. This P' is also a proof of A. Then we substitute $\mathbf{a}_{2m}$ for all the occurrences of d_m in P' for all $m = 0, 1, ...$, and let P'' be the tree obtained from P' by the substitution. Then P'' is a proof of A in $\mathcal{P}$. ■

Lemma A.11. For any $A \in \mathcal{P}$ and $w \in W$, $(\mathcal{M}, \sigma, w) \models A$ if and only if $(\mathcal{M}^d, \sigma, w) \models A$.

Now, we complete the proof of the completeness by showing that for any $A \in \mathcal{P}$, if $\nvdash_{\mathrm{QCY}} A$, then $(\mathcal{M}, \sigma', w) \nvDash A$ for some world $w \in W$ and some assignment σ'. Let $A \in \mathcal{P}$. Suppose $\nvdash_{\mathrm{QCY}} A$. Let A^c be the universal closure of A. Then $\nvdash^*_{\mathrm{QCY}} A^c$. This means $[\![A^c]\!] \neq \mathbf{1}$. By Lemma A.3, there is a $w \in \mathcal{F}_{(\mathcal{Q}_1, \mathcal{Q}_2)}(\mathbb{L})$ such that $[\![A^c]\!] \notin w$. By Lemma A.9, we have $(\mathcal{M}^d, \sigma, w) \nvDash A^c$. Then there is another assignment σ' such that $(\mathcal{M}^d, \sigma', w) \nvDash A$. Since $A \in \mathcal{P}$, we have $(\mathcal{M}, \sigma', w) \nvDash A$. ■

Remark A.12. To prove the completeness of QGL_ω (Theorem 6.2), we modify the above proof as follows: We denote, by $\mathcal{E}^\omega(\mathcal{P}^d)$, the set of formulae by obtained by adding the constants $\mathbf{d}_0, \mathbf{d}_1, ...$ to the original language. Let A be a formula in $\mathcal{E}^\omega(\mathcal{P})$. We consider the set of all closed subformula $\mathcal{CS}^*(A)$ instead of $\mathcal{CP}^*$. Then we can modify the above proof for the completeness of QCY to that of QGL_ω.

Appendix B: Equality and Global Interpretations in Models

It would be natural to include equality in predicate common knowledge logics for application purposes. However, the inclusion of equality to predicate epistemic (more generally modal) logics raises some subtle problems (cf., Garson [3] and Fagin *et al.* [1]). When we interpret equality as the *real* equality independent of possible worlds in Kripke models, the equality would become related closely to common knowledge. More generally, global interpretations of predicate symbols are related to common knowledge, where interpretations of predicate symbols independent of a possible world are called *global*. In this appendix, we consider global interpretations of predicate symbols, in particular, equality, in our logic QCY as well as their relationships to common knowledge both in syntactical and semantic treatments.

We take QCY as our basic logic and discuss two ways of syntactical treatments of global interpretations of predicate symbols. Recall that all Kripke models are assumed to be serial and transitive.

In Subsection B.1, we give semantical and syntactical preliminaries. We introduce generated Kripke models for QCY and show that it suffices to consider generated ones only. Next we discuss two ways of extending formal system QCY by adding some axioms and show relationship between them. In Subsection B.2, we discuss general treatment of global interpretations of predicate symbols. In Subsection B.3, we present a theorem which descibes the completeness of QCY when we restrict our models with the real equality.

B.1 Preliminaries

We say that a Kripke frame $\mathcal{F} = (W; R_1, ..., R_n; M)$ *is generated* iff there is a world $w \in W$ such that every $u \in W$ is reachable from w. We say that a Kripke model $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ *is generated* iff $\mathcal{F}$ is generated. Recall that u *is reachable from* w iff there are $w_0 = w_1, w_2, ..., w_m = u$ in W such that for each $k = 0, 1, ..., k \Leftrightarrow 1$, $w_k R_i w_{k+1}$ for some $i \in N$.

Lemma B.1. Let A be any formula. Then $\mathcal{M} \models A$ for all Kripke models $\mathcal{M}$ if and only if $\mathcal{M} \models A$ for all generated Kripke models $\mathcal{M}$.

Proof. The *if*-part is essential. Suppose that $\mathcal{M} \not\models A$ for some $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ with $\mathcal{F} = (W; R_1, ..., R_n; M)$. There is a $w \in W$ and an assignment σ such that $(\mathcal{M}, \sigma, w) \not\models A$. Let $W' = \{u \in W : u$ is reachable from $w\}$, and $R_1', ..., R_n'$ the restrictions of $R_1, ..., R_n$ to W'. Then $\mathcal{F}' = (W'; R_1', ..., R_n'; M)$ is a generated Kripke frame. Let $\mathcal{I}'$ be the restriction of $\mathcal{I}$ to $\mathcal{F}'$. Then $\mathcal{M}' = (\mathcal{F}', \mathcal{I}')$ is a generated Kripke model and $(\mathcal{M}', \sigma, w) \not\models A$. I.e., $\mathcal{M}' \not\models A$. ■

Hereafter, all Kripke models in this Appendix are assumed to be generated.

We discuss extensions of formal system QCY by adding axioms. As a starting point, we recall the situation in QKD4^n. Let us take an arbitrary set Σ of closed formulae. We have two ways of extending QKD4^n with Σ. One way is to include Σ as logical axioms, i.e., formulae in Σ are allowed to be initial formulae in proofs. In this case, the extended system is denoted as QKD$4^n(\Sigma)$. We write $\vdash_{\mathrm{QKD4}^n(\Sigma)} A$ iff A is provable in QKD$4^n(\Sigma)$. The other way is to consider *non-logical axioms* (or *theory*). That is, we define $\Sigma \vdash_{\mathrm{QKD4}^n} A$ iff $\vdash_{\mathrm{QKD4}^n} \bigwedge_{i=1}^k D_i \supset A$ for some finitely many $D_1, ..., D_k \in \Sigma$. However, $\Sigma \vdash_{\mathrm{QKD4}^n} A$ is not enough to capture $\vdash_{\mathrm{QKD4}^n(\Sigma)} A$. In fact, we need to strengthen Σ into $\mathrm{C}^*(\Sigma) := \{\mathrm{B}_e(D) : D \in \Sigma$ and $e \in N^*\}$. Then we have the following theorem connecting these two ways of introducing nonlogical axioms, which is known as the deduction theorem.

Theorem B.2 (Deduction Theorem for QKD4^n). Let Σ be any set of closed formulae and A any formula. Then $\mathrm{C}^*(\Sigma) \vdash_{\mathrm{QKD4}^n} A$ if and only if $\vdash_{\mathrm{QKD4}^n(\Sigma)} A$.

Now, we consider these two ways of extensions in the case of QCY. We have to pay attentions to the fact that QCY allow infinitary proofs (and the fact that we have only completeness for QCY but not strong completeness). Hence, we restrict Σ to be a finite set, or equivalently to be a singleton set $\{D\}$. We write QCY(Σ) for the this extension to allow formulae in Σ to be initial formulae in proofs, whose provability is denoted by $\vdash_{\mathrm{QCY}(\Sigma)}$. The other way is to define $\Sigma \vdash_{\mathrm{QCY}} A$ by $\vdash_{\mathrm{QCY}} \bigwedge_{i=1}^k D_i \supset A$ for some finitely many $D_1, ..., D_k \in \Sigma$. Similar to the case of QKD4^n, we need to strengthen Σ. Here, the set $\mathrm{C}^*(\Sigma)$ is infinite even if Σ consists of one formula. However, we can use the common knowledge operator C in QCY, rather than the above C^*.

Lemma B.3 (Deduction Theorem for QCY). Let Σ be a finite set of closed formulae, and A a formula. Then $\mathrm{C}(\Sigma) \vdash_{\mathrm{QCY}} A$ if and only if $\vdash_{\mathrm{QCY}(\Sigma)} A$.

Proof. The *only-if* part follows from the fact that $\vdash_{\mathrm{QCY}(\Sigma)} \mathrm{C}(D)$ for any $D \in \Sigma$. We show the *if* part. For the sake of simplicity, we assume that Σ is a singleton set $\{D\}$. First, we claim that for every proof P in QCY and every formula E in P, it holds that $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \supset E$. We show this by induction on P. If P consists of only of an initial formula E other than D, then $\mathrm{C}(D) \supset E$ is provable by L1 and MP. If P consists only of D, then $\mathrm{C}(D) \supset D$ is provable in QCY. If P is non-trivial, then we have only to show that $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \supset E$ for its lowermost formula E. We divide the cases depending on the last inference rule which derives E from its premise(s). We consider only the Necessitation and CI*.

Necessitation: The E is of the form $\mathrm{B}_i(F)$ for some F. By the induction hypothesis, we have $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \supset F$. Then $\vdash_{\mathrm{QCY}} \mathrm{B}_i\mathrm{C}(D) \supset \mathrm{B}_i(E)$. Since $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \supset \mathrm{B}_i\mathrm{C}(D)$, we have $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \supset \mathrm{B}_i(F)$.

CI* : The E is of the form $F_1 \supset T(\mathrm{C}(F_2))$ for some F_1 and F_2. By the induction hypothesis, we have $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \supset (F_1 \supset T(\mathrm{B}_e(F_2)))$ for all $e \in N^*$. Hence, $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \wedge F_1 \supset T(\mathrm{B}_e(F))$ for all $e \in N^*$. Hence $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \wedge F_1 \supset T(\mathrm{C}(F_2))$. I.e., $\vdash_{\mathrm{QCY}} \mathrm{C}(D) \supset (F_1 \supset T(\mathrm{C}(F_2)))$. ■

We have the following semantical counterpart of Lemma B.3, which is easily shown. For a finite nonempty set of formulae $\Phi = \{A_1, ..., A_l\}$, we denote $A_1 \wedge ... \wedge A_l$ by $\bigwedge \Phi$. In the following, we write $\mathcal{M} \models \Phi$ for $\mathcal{M} \models \bigwedge \Phi$.

Lemma B.4. Let Σ be a finite set of closed formulae and $A \in \mathcal{P}$. Then $\mathcal{M} \models \bigwedge \mathrm{C}(\Sigma) \supset A$ for any Kripke models $\mathcal{M}$ if and only if $\mathcal{M} \models A$ for any Kripke models $\mathcal{M}$ with $\mathcal{M} \models \Sigma$.

By the Completeness Theorem for QCY, we have the following.

Lemma B.5. Let Σ be a finite set of closed formulae and $A \in \mathcal{P}$. Then the following four statements are equivalent:

(1): $\vdash_{\mathrm{QCY}(\Sigma)} A$;

(2): $\mathrm{C}(\Sigma) \vdash_{\mathrm{QCY}} A$;

(3): $\mathcal{M} \models \bigwedge \mathrm{C}(\Sigma) \supset A$ for any Kripke models $\mathcal{M}$;

(4): $\mathcal{M} \models A$ for any Kripke models $\mathcal{M}$ with $\mathcal{M} \models \Sigma$.

B.2 Global Interpretations of Predicate Symbols

In this subsection, we fix a particular predicate symbol $\mathbf{P}$ with arity m. We say that $\mathbf{P}$ is *global* in a Kripke model $\mathcal{M} = (\mathcal{F}, \mathcal{I}) = ((W; R_1, ..., R_n; M), \mathcal{I})$ iff the interpretation $\tilde{P}^w$ of $\mathbf{P}$ in $\mathcal{M}$ is independent of $w \in W$, i.e.,

$$\tilde{P}^w(a_1, ..., a_m) = \tilde{P}^u(a_1, ..., a_m) \text{ for all } w, u \in W \text{ and } (a_1, ..., a_m) \in M^m. \tag{B.1}$$

Now, we consider the syntactical counterpart of (B.1). We denote the following formula by $Gl(\mathbf{P})$:

$$\begin{aligned} &\forall x_1 ... \forall x_m (\mathbf{P}(x_1, ..., x_m) \supset \mathrm{C}(\mathbf{P}(x_1, ..., x_m))) \wedge \\ &\forall x_1 ... \forall x_m (\neg \mathbf{P}(x_1, ..., x_m) \supset \mathrm{C}(\neg \mathbf{P}(x_1, ..., x_m))) \end{aligned} \tag{B.2}$$

That is, if $\mathbf{P}(\mathbf{a}_1, ..., \mathbf{a}_m)$ or its negation holds, then it or its negation is common knowledge among the agents. The syntactical counterpart of (B.1) is, in fact, the common knowledge $\mathrm{C}(Gl(\mathbf{P}))$ of $Gl(\mathbf{P})$.[5] First, we state the following lemma.

Lemma B.6. Let $\mathcal{M}$ be a Kripke model. Then the following three statements are equivalent:

[5] In this paper, we assume that the interpretations of function symbols are always global in Kripke models. Wolter [21] discussed some problems when functions symbols are interpreted locally and/or globally.

(1): $\mathcal{M}$ satisfies (B.1);

(2): $\mathcal{M} \vDash \mathrm{C}(Gl(\mathbf{P}))$;

(3): $\mathcal{M} \vDash Gl(\mathbf{P})$.

Proof. We show here that (1) implies (3). Let $\mathcal{M} = (\mathcal{K}, \mathcal{I}) = ((W; R_1, ..., R_n; M), \mathcal{I})$. Let w be a world in W and σ an assignment. Suppose $(\mathcal{M}, \sigma, w) \vDash \mathbf{P}(\mathbf{a}_1, ..., \mathbf{a}_m)$. By (B.1), we have $(\mathcal{M}, \sigma, u) \vDash \mathbf{P}(a_1, ..., a_m)$ for every u reachable from w. Hence $(\mathcal{M}, \sigma, w) \vDash \mathrm{C}(\mathbf{P}(\mathbf{a}_1, ..., \mathbf{a}_m))$. Since σ is arbitrary, we have

$$(\mathcal{M}, \sigma, w) \vDash \forall x_1 ... \forall x_m (\mathbf{P}(x_1, ..., x_m) \supset \mathrm{C}(\mathbf{P}(x_1, ..., x_m)))$$

for every σ. Similarly, we have

$$(\mathcal{M}, \sigma, w) \vDash \forall x_1 ... \forall x_m (\neg\mathbf{P}(x_1, ..., x_m) \supset \mathrm{C}(\neg\mathbf{P}(x_1, ..., x_m)))$$

for every σ. Hence $(\mathcal{M}, \sigma, w) \vDash Gl(\mathbf{P})$. ■

The validity in the class of Kripke models satisfying (B.1) is captured by the extension QCY or by assuming the common knowledge $\mathrm{C}(Gl(\mathbf{P}))$ of $Gl(\mathbf{P})$ as a nonlogical axiom in QCY.

Theorem B.7 (Global Interpretation of P). For any $A \in \mathcal{P}$, the following three conditions are equivalent:

(1): $\vdash_{\mathrm{QCY}(Gl(\mathbf{P}))} A$;

(2): $\mathrm{C}(Gl(\mathbf{P})) \vdash_{\mathrm{QCY}} A$;

(3): $\mathcal{M} \vDash A$ for all Kripke models $\mathcal{M}$ satisfying (B.1).

Proof. Lemma B.3 implies the equivalence between (1) and (2). From Lemmas B.6 and B.4, it follows that (3) is equivalent to that $\mathcal{M} \vDash \mathrm{C}(Gl(\mathbf{P})) \supset A$ for all Kripke models $\mathcal{M}$. By the completeness of QCY, this is equivalent to (2). ■

B.3 Global Equality and Normal Models

Now, let us consider the problem of equality. Let $\mathcal{L} = [\mathbf{f}_0, \mathbf{f}_1, ...; \mathbf{P}_0, \mathbf{P}_1, ...,]$ have finitely many symbols and contain an equality symbol $\equiv$. In this section, when the interpretation $\dot{\equiv}^w$ of $\equiv$ at the world w is required to be not only global but also the standard identity on the domains of Kripke models, we see what axioms are required correspondingly for QCY.

We say that $\mathcal{M} = ((W; R_1, ..., R_n; M), \mathcal{I})$ is *normal* iff the interpretation $\dot{\equiv}^w$ of the binary relation $\equiv$ at any w in $\mathcal{M}$ is an identity relation, i.e., $a \dot{\equiv}^w b$ if and only if a and b are identical. This implies that in a normal model $\mathcal{M}$, the interpretation of $\equiv$ is global. Hence, Lemma B.6 is applied to equality symbol $\equiv$. We write the corresponding axiom $Gl(\equiv)$ as Eg:

Eg: $\forall x_1 \forall x_2 (x_1 \equiv x_2 \supset \mathrm{C}(x_1 \equiv x_2)) \wedge \forall x_1 \forall x_2 (\neg x_1 \equiv x_2 \supset \mathrm{C}(\neg x_1 \equiv x_2))$.

For the sake of simplicity, we regard Eg as the set consisting only of this formula.

Let us recall the equality axioms:

Reflexivity: $\forall x\,(x \equiv x)$;

Symmetry: $\forall x \forall y\,(x \equiv y \supset y \equiv x)$;

Transitivity: $\forall x \forall y \forall z\,(x \equiv y \wedge y \equiv z \supset x \equiv z)$;

Substitutability: $\forall x_1 ... \forall x_m \forall y_k\,(x_k \equiv y_k \supset f(x_1, ..., x_k, ..., x_m) \equiv f(x_1, ..., y_k, ..., x_m))$;

$\forall x_1 ... \forall x_m \forall y_k\,(x_k \equiv y_k \supset (P(x_1, ..., x_k, ..., x_m) \supset P(x_1, ..., y_k, ..., x_m)))$,

where f and P are m-ary function symbol and predicate symbol for m $(1 \leq m < \omega)$ and $k = 1, ..., m$. We denote the set of these axioms by Eq. Note that Eq is finite, since $\mathcal{L}$ is finite.

Lemma B.8. Let $\mathcal{M}$ be a normal Kripke model. Then $\mathcal{M} \models$ Eq$\cup$Eg.

As in Theorem B.7, we have two ways of introducing the equality axioms to QCY. The next theorem connects these treatments and normal models.

Theorem B.9 (Treatment of Equality). Let A be a formula. Then the following four statements are equivalent:

(1): $\vdash_{\mathrm{QCY(Eq\cup Eg)}} A$;

(2): $\mathrm{C(Eq\cup Eg)} \vdash_{\mathrm{QCY}} A$;

(3): $\mathcal{M} \models A$ for any Kripke models $\mathcal{M}$ with $\mathcal{M} \models$ Eq$\cup$Eg;

(4): $\mathcal{M} \models A$ for any normal Kripke models $\mathcal{M}$.

Proof. By Lemma B.5, (1),(2) and (3) are all equivalent. By Lemma B.8, (3) implies (4). We have only to show that (4) implies (3). It suffices to prove that for each Kripke model $\mathcal{M}$ with $\mathcal{M} \models$ Eq$\cup$Eg, there is a normal Kripke model $\mathcal{M}'$ such that $\mathcal{M} \models A$ if and only if $\mathcal{M}' \models A$. The construction of such an $\mathcal{M}'$ is now shown below.

Lemma B.10. Let $\mathcal{M} = ((W; R_1, ..., R_n; M), \mathcal{I})$ be a Kripke model with $\mathcal{M} \models$ Eq$\cup$Eg. Let $\tilde{\equiv}^w$ be the interpretation of $\equiv$ in world w in $\mathcal{M}$. Then $\tilde{\equiv}^w$ is an equivalence relation on M for any $w \in W$. Moreover, $\tilde{\equiv}^w$ is independent of w.

The last independence claim of Lemma B.10 needs the assumption that $\mathcal{M}$ is a generated model. However, Lemma B.1 guarantees that we can focus only on generated models.

Let $\mathcal{M} = ((W; R_1, ..., R_n; M), \mathcal{I})$ be a Kripke model with $\mathcal{M} \models$ Eq$\cup$Eg. Then Lemma B.10 states that equivalence relations $\tilde{\equiv}^w$ are the same over W. Hence we denote $\tilde{\equiv}^w$ by $\tilde{\equiv}$. Define the set of equivalence classes $M/\tilde{\equiv}$. We denote the equivalence class including a by $[a]$. The following lemma guarantees that we can treat any elements in $[a]$ as the same as a in $\mathcal{M}$.

Lemma B.11. Let $\mathcal{M} = ((W; R_1, ..., R_n; M), \mathcal{I})$ be a Kripke model with $\mathcal{M} \models$ Eq$\cup$Eg. Then the following hold for any $w \in W$:

(1): For any m-ary function symbol f, if $a_k \tilde{\equiv} b_k$ for $k = 1, ..., m$, then $\tilde{f}(a_1, ..., a_m)$ $\tilde{\equiv}$ $\tilde{f}(b_1, ..., b_m)$;

(2): For any m-ary predicate symbol P, if $a_k \tilde{\equiv} b_k$ for $k = 1, ..., m$, then $\tilde{P}^w(a_1, ..., a_m) =$ $\top$ if and only if $\tilde{P}^w(b_1, ..., b_m) = \top$.

Now, we define $\mathcal{M}' = (\mathcal{F}', \mathcal{I}')$ by:

N1: $\mathcal{F}'$ is obtained from $\mathcal{F}$ by replacing M by M', i.e., $\mathcal{F}' = (W; R_1, ..., R_n; M')$ with $M' = M/\tilde{\equiv}$;

N2: $\mathcal{I}'(w) = [\tilde{f}'_0, \tilde{f}'_1, ...; \tilde{P}'^w_0, \tilde{P}'^w_1, ...]$ is defined as follows:

N2-1: for each m-ary function symbol f,

$\tilde{f}'([a_1], ..., [a_m]) = [\tilde{f}(a_1, ..., a_m)]$ for all $[a_1], ..., [a_m] \in M'$;

N2-2: for each m-ary predicate symbol P,

$\tilde{P}'^w([a_1], ..., [a_m]) = \tilde{P}^w(a_1, ..., a_m)$ for all $[a_1], ..., [a_m] \in M'$.

Lemma B.11 guarantees the well-definedness of $\mathcal{I}'$. This $\mathcal{M}' = (\mathcal{F}', \mathcal{I}')$ is a normal Kripke model. Let $\sigma : FV \rightarrow M$ be an arbitrary assignment on $\mathcal{M}$. A mapping $\sigma' : FV \rightarrow M'$ defined by

$$\sigma'(\mathbf{a}) = [\sigma(\mathbf{a})] \text{ for any } a \in FV.$$

is an assignment on $\mathcal{M}'$. Conversely, for assignment $\tau : FV \rightarrow M$ on $\mathcal{M}'$, there is an assignment σ on $\mathcal{M}$ with $\tau = \sigma'$. Then we have the following.

Lemma B.12. (1): For any term t, $V(t, (\mathcal{I}', \sigma')) = [V(t, (\mathcal{I}, \sigma))]$;

(2): For any $w \in W$ and any $B \in \mathcal{P}$, $(\mathcal{M}, \sigma, w) \models B$ if and only if $(\mathcal{M}', \sigma', w) \models B$.

By this lemma, we have a normal Kripke model $\mathcal{M}'$ such that $\mathcal{M} \models A$ if and only if $\mathcal{M}' \models A$. It follows that (4) implies (3). This completes the proof of Theorem B.9. ■

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